

Algebra

A-Level Maths 9709 · Topic 3.1 revision notes

Everything from the 4 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. The modulus function

3. Partial fractions

2. Polynomial division, factor and remainder theorems

4. The binomial series for any rational n

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1 The modulus function

Syllabus 3.1

$|x|$ is the distance of x from 0. Two facts do most of the work: $|a| = |b| \iff a^2 = b^2$, and $|x - a| < b \iff a - b < x < a + b$. When one side is NOT a modulus, any answer found by splitting into cases must be checked in the original equation.

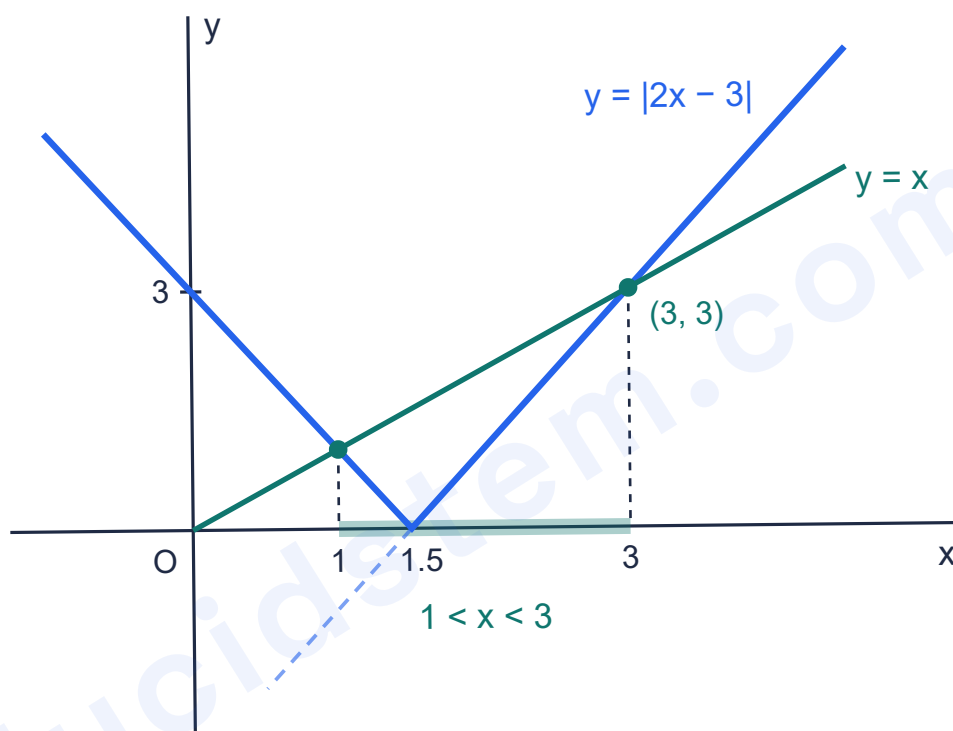


Fig. 1: $y = |2x - 3|$ is $y = 2x - 3$ with the part below the x -axis (dashed) reflected up. It meets $y = x$ at $x = 1$ and $x = 3$, so $|2x - 3| < x$ for $1 < x < 3$

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Worked example: Sketching a modulus graph

Sketch $y = |2x - 3|$, stating the coordinates of the points where it meets the axes.

1. Vertex: $2x - 3 = 0$ gives $x = 1.5$, so the V touches the x -axis at $(1.5, 0)$.
2. y -intercept: $x = 0$ gives $y = |-3| = 3$, so $(0, 3)$.
3. For $x \geq 1.5$ the graph is $y = 2x - 3$; for $x < 1.5$ it is the reflection $y = 3 - 2x$.

Answer: A V with vertex $(1.5, 0)$ and y -intercept $(0, 3)$, arms of gradient 2 and -2 (Fig. 1).

Choose the method from the shape of the question.

- ✓ **Sketch:** vertex where $ax + b = 0$, y -intercept $|b|$, arms of gradient $\pm a$.
- ✓ **$|f| = |g|$:** square both sides, or solve $f = g$ and $f = -g$.
- ✓ **$|f| = \text{line}$:** split into cases and **CHECK** each root in the original equation.
- ✓ **$|x - a| < b$:** $a - b < x < a + b$; for $>$ take the two outside pieces.
- ✓ **$|f| < |g|$:** square, then solve the quadratic inequality.
- ✓ **Modulus against a line:** sketch, find valid crossings, read off the region.

Watch out!

The vertex of $y = |2x - 3|$ is at $x = 1.5$, not $x = 3$ or $x = -1.5$. Solve $ax + b = 0$ properly; the most common slip is forgetting to divide by a .

In Case 2, $-(2x - 1)$ is $-2x + 1$, not $-2x - 1$. Put brackets round the whole side before you change its sign.

Never square when one side might be negative: $2x + 5 < |x + 1|$ squared gives a wrong answer, because squaring a negative left side flips which side is bigger. Square only when both sides are moduli (or are known to be non-negative).

2 Polynomial division, factor and remainder theorems

Syllabus 3.1

$p(x) = (\text{divisor}) \times (\text{quotient}) + \text{remainder}$, where the remainder has lower degree than the divisor. The remainder when $p(x)$ is divided by $(ax + b)$ is $p\left(-\frac{b}{a}\right)$, and $(ax + b)$ is a factor exactly when $p\left(-\frac{b}{a}\right) = 0$.

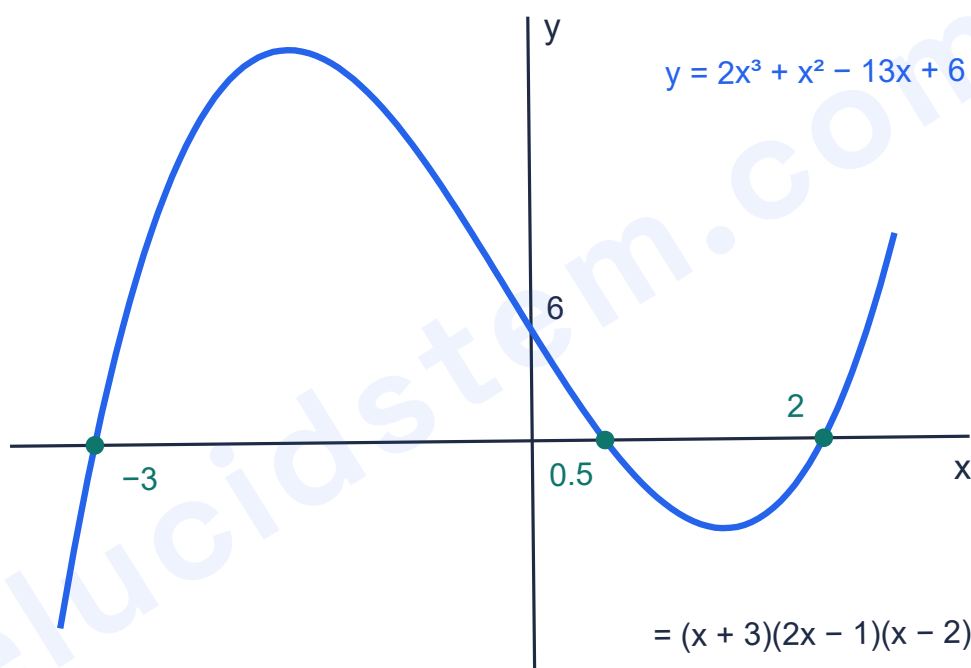


Fig. 1: $p(x) = 2x^3 + x^2 - 13x + 6$ crosses the x -axis at -3 , 0.5 and 2 , so $(x + 3)$, $(2x - 1)$ and $(x - 2)$ are its factors

Worked example: Dividing by a linear divisor

Find the quotient and remainder when $3x^3 - 4x^2 + 5x - 7$ is divided by $(x - 2)$.

1. $3x^3 \div x = 3x^2$. Multiply back: $3x^2(x - 2) = 3x^3 - 6x^2$. Subtract: $2x^2 + 5x$.
2. $2x^2 \div x = 2x$. Multiply back: $2x(x - 2) = 2x^2 - 4x$. Subtract: $9x - 7$.
3. $9x \div x = 9$. Multiply back: $9(x - 2) = 9x - 18$. Subtract: 11 .
4. Check with the remainder theorem: $p(2) = 24 - 16 + 10 - 7 = 11$. It agrees.

Answer: Quotient $3x^2 + 2x + 9$, remainder 11

Substitute the root of the divisor.

- ✓ **Divide:** leading term, multiply back, subtract, repeat; include zero terms for missing powers.
- ✓ **Stop:** when the remainder's degree is below the divisor's degree.
- ✓ **Remainder theorem:** remainder on dividing by $(ax + b)$ is $p\left(-\frac{b}{a}\right)$.
- ✓ **Factor theorem:** $(ax + b)$ is a factor if and only if $p\left(-\frac{b}{a}\right) = 0$.
- ✓ **Unknowns:** one equation per condition, then solve simultaneously.

Watch out!

Subtracting a negative term is where marks go: $(2x^2 + 5x) - (2x^2 - 4x) = 9x$, not x . Put the line you subtract in brackets and change every sign.

For the divisor $(x + 3)$ you substitute $x = -3$, not $x = 3$; for $(2x - 1)$ you substitute $x = \frac{1}{2}$, not $x = 1$ or $x = 2$.

A remainder condition is $p(c) = r$, not $p(c) = 0$. Only a factor gives zero.

3 Partial fractions

Syllabus 3.1

Choose the form from the denominator, multiply through to get an identity in x , then find the constants by substituting the root of each linear factor and comparing coefficients for whatever is left. A repeated factor $(cx + d)^2$ needs BOTH $\frac{B}{cx + d}$ and $\frac{C}{(cx + d)^2}$; a quadratic factor $cx^2 + d$ needs a numerator $Bx + C$. If the numerator has the same degree as the denominator, include a constant term as well.

Worked example: Two distinct linear factors

Express $\frac{8x - 1}{(x + 1)(2x - 1)}$ in partial fractions.

1. Form: $\frac{8x - 1}{(x + 1)(2x - 1)} \equiv \frac{A}{x + 1} + \frac{B}{2x - 1}$.
2. Multiply by the denominator: $8x - 1 \equiv A(2x - 1) + B(x + 1)$.
3. Put $x = -1$ (kills the B term): $-9 = A(-3)$, so $A = 3$.
4. Put $x = \frac{1}{2}$ (kills the A term): $3 = B\left(\frac{3}{2}\right)$, so $B = 2$.
5. Check at $x = 0$: left side $\frac{-1}{-1} = 1$; right side $3 + \frac{2}{-1} = 1$. Agrees.

Answer: $\frac{3}{x + 1} + \frac{2}{2x - 1}$

Form, identity, substitute, compare.

- ✓ **Factorise** the denominator fully, and compare degrees: equal degrees need a constant term A as well.
- ✓ **Form:** one fraction per linear factor; $\frac{B}{cx + d} + \frac{C}{(cx + d)^2}$ for a repeated factor; $\frac{Bx + C}{cx^2 + d}$ for a quadratic factor.
- ✓ **Identity:** multiply through by the whole denominator.
- ✓ **Substitute** the root of each linear factor to find its constant directly.
- ✓ **Compare coefficients** (usually x^2 and constants) for the rest, then check one more coefficient.

Watch out!

Factorise the denominator before choosing the form. $x^2 - 2x - 3$ is not a 'quadratic factor' needing $Bx + C$; it is $(x - 3)(x + 1)$. Only a quadratic that does not factorise, such as $x^2 + 3$, gets the $Bx + C$ numerator.

When substituting $x = -\frac{1}{2}$, every bracket must be evaluated at $-\frac{1}{2}$:

$(x - 1) = -\frac{3}{2}$, not -1 . A single slip here makes C wrong, so always check with a coefficient comparison.

Writing only $\frac{A}{x + 1} + \frac{C}{(2x + 1)^2}$ loses the middle term. With two unknowns the identity cannot hold for every x , and you will find the 'answers' disagree when you check.

$\frac{B}{x^2 + 3}$ with a constant numerator is the wrong form: it gives too few unknowns, and the x coefficients will not match. Over an irreducible quadratic always write $Bx + C$.

Leaving out the constant term is the classic error here. Without it the identity $2x^2 + x + 3 \equiv B(2x + 1) + C(x - 1)$ is impossible: the right side has no x^2 term. Compare the degrees of the numerator and denominator before you write down the form.

4 The binomial series for any rational n

Syllabus 3.1

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots, \text{ valid for } |x| < 1$$

when n is not a positive integer. For $(a+bx)^n$, first take out a^n : $a^n \left(1 + \frac{b}{a}x\right)^n$, valid for $\left|\frac{bx}{a}\right| < 1$.

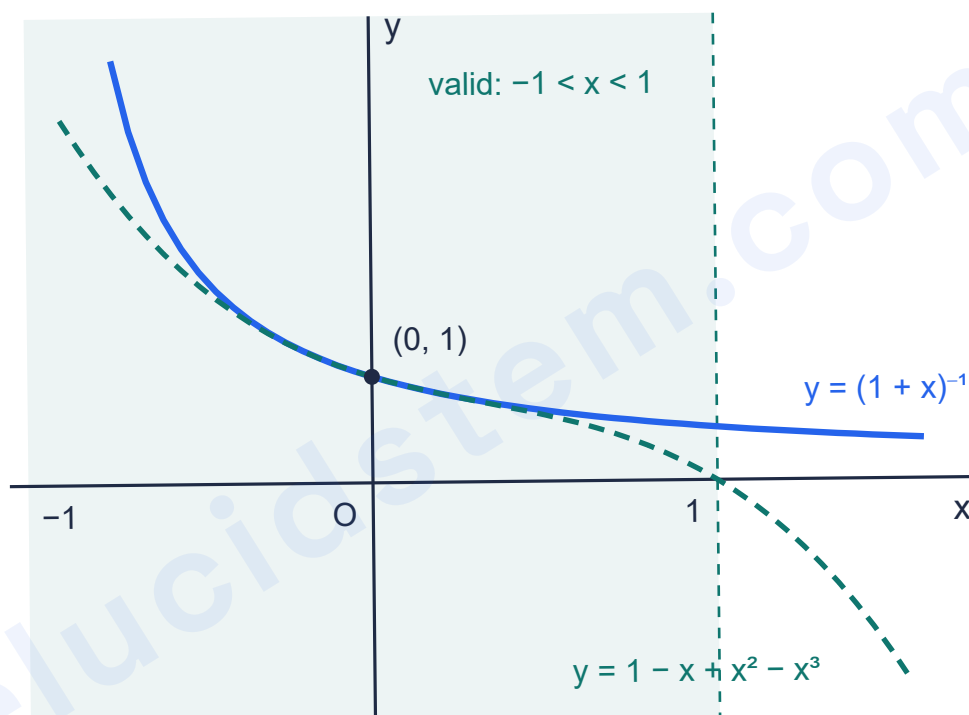


Fig. 1: near $x = 0$ the first four terms of the series for $(1+x)^{-1}$ (dashed) hug the curve; beyond $x = 1$ the series is not valid and the approximation falls away

Worked example: A negative index

Expand $(1+x)^{-2}$ in ascending powers of x , up to and including the term in x^3 .

1. $n = -2$: coefficient of x is -2 .

2. Coefficient of x^2 : $\frac{(-2)(-3)}{2} = 3$.

3. Coefficient of x^3 : $\frac{(-2)(-3)(-4)}{6} = -4$.

Answer: $1 - 2x + 3x^2 - 4x^3$, valid for $|x| < 1$

Make it $(1 + \dots)^n$, then expand.

✓ **Standard form:** $(a + bx)^n = a^n \left(1 + \frac{b}{a}x\right)^n$.

✓ **Coefficients:** $1, n, \frac{n(n-1)}{2!}, \frac{n(n-1)(n-2)}{3!}, \dots$

✓ **Brackets:** raise the whole term, sign and number, to each power.

✓ **Validity:** $\left|\frac{bx}{a}\right| < 1$; for sums take the narrowest range.

✓ **Products:** collect every pair of terms that multiply to the power you need.

Watch out!

$(-4x)^2 = 16x^2$, not $-4x^2$ or $4x^2$. Write the bracket in every term, then square or cube both the number and the sign.

$(2 - x)^{-3}$ is NOT

$2 \left(1 - \frac{x}{2}\right)^{-3}$. The factor taken out is $2^{-3} = \frac{1}{8}$; it carries the same power as the bracket.

Write $\frac{2}{2x - 1}$ as $-2(1 - 2x)^{-1}$ before expanding: the standard series needs the constant term in the bracket to be +1. Expanding $(2x - 1)^{-1}$ 'as if' it were $(1 + 2x)^{-1}$ gives the wrong signs.