

Complex numbers

A-Level Maths 9709 · Topic 3.9 revision notes

Everything from the 4 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Complex numbers: arithmetic in Cartesian form
2. The Argand diagram, modulus-argument and exponential form
3. Complex roots and square roots
4. Loci in the Argand diagram

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1 Complex numbers: arithmetic in Cartesian form

Syllabus 3.9

A complex number is $z = x + iy$ with x, y real and $i^2 = -1$; $\text{Re } z = x$, $\text{Im } z = y$, and the conjugate is $z^* = x - iy$. Two complex numbers are equal only if their real parts are equal AND their imaginary parts are equal. To divide, multiply top and bottom by the conjugate of the bottom.

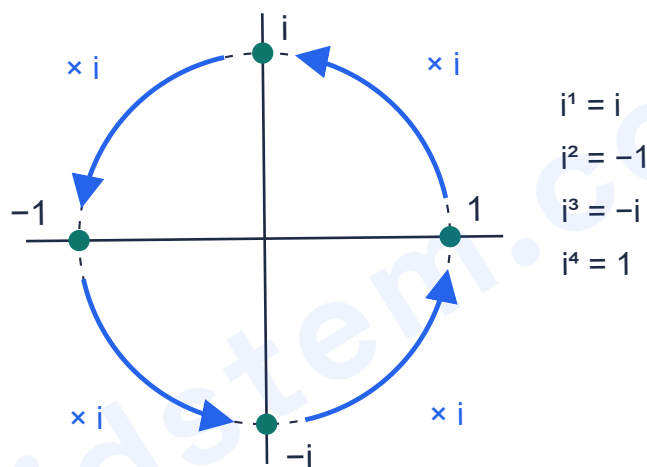


Fig. 1: each multiplication by i moves one step round the cycle $1, i, -1, -i$, so the powers of i repeat every four

Worked example: Quadratics with complex roots

Solve (a) $z^2 + 9 = 0$ (b) $z^2 - 4z + 13 = 0$.

1. (a) $z^2 = -9$, so $z = \pm\sqrt{-9} = \pm\sqrt{9}\sqrt{-1} = \pm 3i$.

2. (b) Quadratic formula: $z = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2}$.

3. $\sqrt{-36} = 6i$, so $z = \frac{4 \pm 6i}{2} = 2 \pm 3i$.

4. Check $2 + 3i$: $(2 + 3i)^2 = 4 + 12i - 9 = -5 + 12i$; then $-5 + 12i - 8 - 12i + 13 = 0 \checkmark$.

Answer: (a) $z = 3i$ or $z = -3i$ (b) $z = 2 + 3i$ or $z = 2 - 3i$

Algebra as usual, then $i^2 = -1$.

- ✓ **Add/subtract:** real parts together, imaginary parts together.
- ✓ **Multiply:** expand the brackets, then replace i^2 by -1 .
- ✓ **Divide:** multiply top and bottom by the conjugate of the bottom.
- ✓ **Unknown z :** write $z = x + iy$ (and $z^* = x - iy$), then equate real and imaginary parts.
- ✓ **Quadratics:** a negative discriminant gives a conjugate pair $p \pm qi$.

Watch out!

$\sqrt{-36}$ is $6i$, not -6 or 6 . And $i^2 = -1$ must be used every time it appears: leaving $-9i^2$ as -9 is the most common lost mark in the whole topic.

$zz^* = x^2 + y^2$, always real and never negative. If your answer to $(3 + 4i)(3 - 4i)$ is -7 , you used $i^2 = +1$.

You cannot divide parts separately:

$\frac{3 + 4i}{1 - 2i}$ is NOT $3 - 2i$. Only a REAL denominator can be shared out between the real and imaginary parts.

2 The Argand diagram, modulus-argument and exponential form

Syllabus 3.9

$z = x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$, where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \arg z$ with $-\pi < \theta \leq \pi$. Then $|z_1 z_2| = |z_1| |z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$; for division, divide the moduli and subtract the arguments.

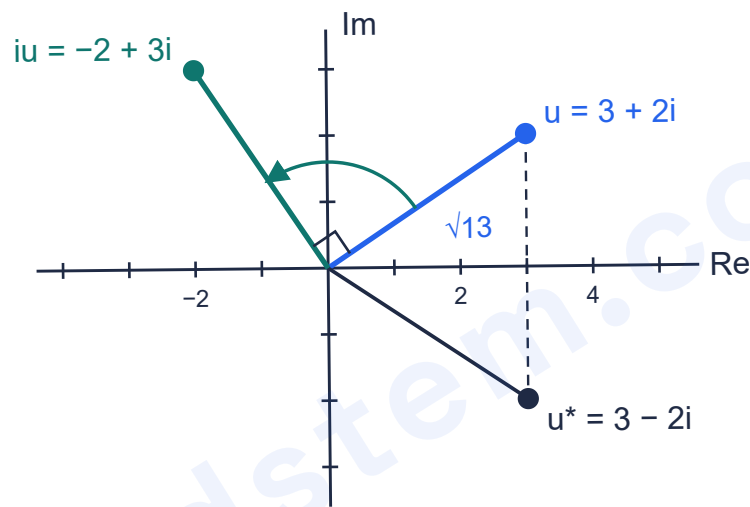


Fig. 1: $u = 3 + 2i$, its conjugate u^* (reflection in the real axis) and $iu = -2 + 3i$ (rotation by 90° anticlockwise about O); all three are $\sqrt{13}$ from O

Worked example: Modulus, argument and three forms

Find the modulus and argument of $z = -1 + \sqrt{3}i$, and write z in polar and exponential form.

1. Sketch: $x < 0$, $y > 0$, so z is in the second quadrant.
2. $|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$.
3. Reference angle $\alpha = \tan^{-1} \frac{\sqrt{3}}{1} = \frac{\pi}{3}$, so $\arg z = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.
4. Polar: $2 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$. Exponential: $2e^{2\pi i/3}$.

Answer: $|z| = 2$, $\arg z = \frac{2\pi}{3}$; $z = 2 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) = 2e^{2\pi i/3}$.

Sketch, then modulus and argument.

- ✓ **Sketch** the point to find its quadrant before using $\tan^{-1}$.
- ✓ **Modulus:** $r = \sqrt{x^2 + y^2}$. **Argument:** adjust the reference angle for the quadrant; $-\pi < \theta \leq \pi$.
- ✓ **Forms:** $x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$.
- ✓ **Multiply:** multiply moduli, add arguments. **Divide:** divide moduli, subtract arguments.
- ✓ **Geometry:** z^* reflects; $+w$ translates; $\times re^{i\theta}$ enlarges and rotates.

Watch out!

Your calculator's $\tan^{-1}\left(\frac{y}{x}\right)$ gives $\tan^{-1}\left(\frac{-4}{-3}\right) = 0.927$ for $-3 - 4i$, which points into the *FIRST* quadrant. The sketch is not optional: it is how you know to adjust.

Moduli **MULTIPLY** but arguments **ADD**. Multiplying the arguments, or adding the moduli, are both common slips, and $\frac{4\pi}{3}$ left unadjusted loses the final mark when the range $-\pi < \theta \leq \pi$ is required.

Multiplying by $-i$ (or dividing by i) rotates **CLOCKWISE**. Check the direction with a simple case: $i \times 1 = i$ moves from the real axis up to the imaginary axis, which is anticlockwise.

3 Complex roots and square roots

Syllabus 3.9

For a polynomial with REAL coefficients, if $a + bi$ is a root then so is $a - bi$, and $(z - (a + bi))(z - (a - bi)) = z^2 - 2az + (a^2 + b^2)$ is a real quadratic factor. To find $\sqrt{p + qi}$, write $(x + iy)^2 = p + qi$ and equate parts: $x^2 - y^2 = p$, $2xy = q$.

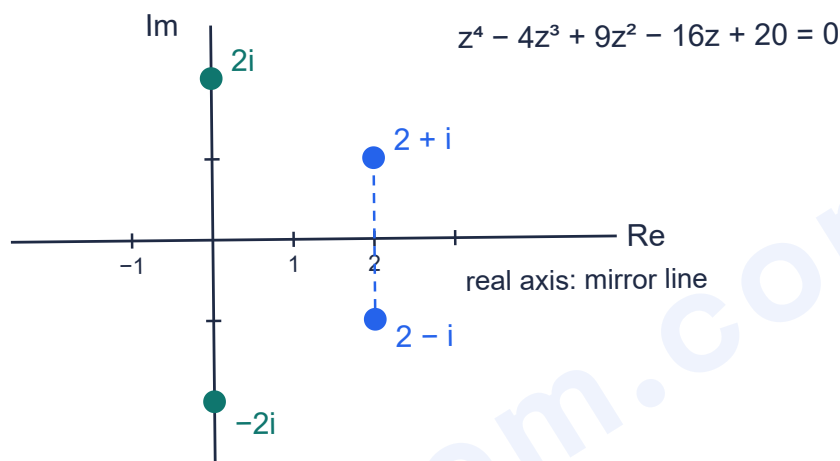


Fig. 1: the four roots of $z^4 - 4z^3 + 9z^2 - 16z + 20 = 0$ are $2 \pm i$ and $\pm 2i$; because the coefficients are real, the roots are symmetric in the real axis

Worked example: A cubic from one complex root

Show that $1 + 2i$ is a root of $z^3 - z^2 + 3z + 5 = 0$, and find the other roots.

1. $(1 + 2i)^2 = 1 + 4i - 4 = -3 + 4i$ and $(1 + 2i)^3 = (1 + 2i)(-3 + 4i) = -3 + 4i - 6i - 8 = -11 - 2i$.
2. Substitute: $(-11 - 2i) - (-3 + 4i) + 3(1 + 2i) + 5 = (-11 + 3 + 3 + 5) + (-2 - 4 + 6)i = 0 \checkmark$.
3. Real coefficients, so $1 - 2i$ is also a root.
4. Quadratic factor: sum = 2, product = $1 + 4 = 5$, giving $z^2 - 2z + 5$.
5. Divide (or compare coefficients): $z^3 - z^2 + 3z + 5 = (z^2 - 2z + 5)(z + 1)$.

Answer: The roots are $1 + 2i$, $1 - 2i$ and -1 .

Pair it, factor it, finish it.

- ✓ **Pair:** real coefficients mean $a + bi$ brings $a - bi$.
- ✓ **Factor:** $z^2 - 2az + (a^2 + b^2)$.
- ✓ **Finish:** divide or compare coefficients for the remaining factor, then solve.
- ✓ **Unknown coefficients:** substitute the root and equate real and imaginary parts.
- ✓ **Square roots:** $x^2 - y^2 = p$, $2xy = q$
; reject $x^2 < 0$; signs of x, y linked by q .

Watch out!

When you substitute a complex root, ALL the real terms go into one equation and ALL the imaginary terms into the other. A single combined equation like $-6 - 3p + q - 2 + 4p + 2q = 0$ throws away information and cannot give two unknowns.

The quadratic factor is $z^2 - (\text{sum})z + \text{product}$. For roots $2 \pm i$ that is $z^2 - 4z + 5$; writing $z^2 + 4z + 5$ (wrong sign) makes the division leave a remainder.

Keep the signs linked: xy negative means x and y have OPPOSITE signs. Writing the roots as $\pm 1 \pm 4i$ (four numbers) is wrong; there are exactly two.

4 Loci in the Argand diagram

Syllabus 3.9

$|z - a| = k$ is the circle with centre a and radius k ; $|z - a| = |z - b|$ is the perpendicular bisector of the points a and b ; $\arg(z - a) = \theta$ is the half-line from a (not including a) at angle θ to the positive real direction. Inequalities give the regions inside or outside these boundaries.

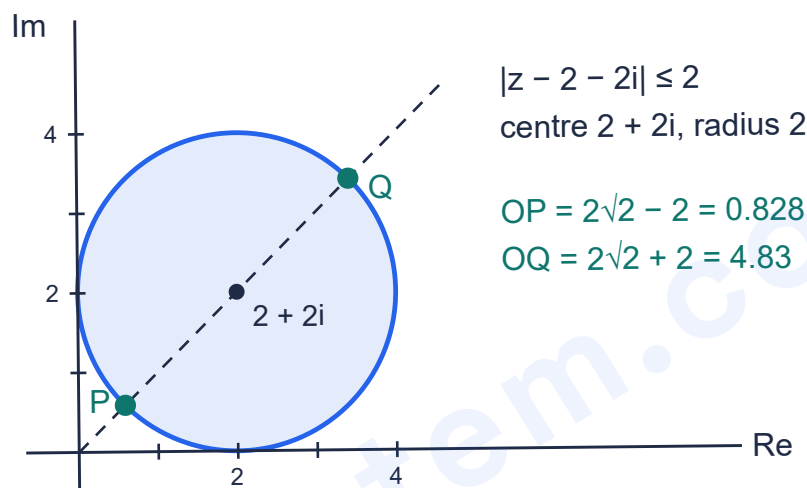


Fig. 1: the disc $|z - 2 - 2i| \leq 2$; the line through O and the centre meets the circle at P (least $|z|$) and Q (greatest $|z|$), and the circle touches both axes, so $\arg z$ runs from 0 to $\pi/2$

Worked example: Centre and radius

Describe the locus $|z - 3 + 2i| = 4$ and find its Cartesian equation.

1. Write it as $|z - (3 - 2i)| = 4$: points at distance 4 from $3 - 2i$.
2. Circle with centre $(3, -2)$ and radius 4.
3. With $z = x + iy$: $|(x - 3) + (y + 2)i| = 4$, so $(x - 3)^2 + (y + 2)^2 = 16$.

Answer: The circle centre $3 - 2i$, radius 4: $(x - 3)^2 + (y + 2)^2 = 16$.

Rewrite as $|z - a|$, then read the geometry.

- ✓ $|z - a| = k$: circle, centre a , radius k ; $<$ inside, $>$ outside.
- ✓ $|z - a| = |z - b|$: perpendicular bisector of a and b .
- ✓ $\arg(z - a) = \theta$: half-line from a (excluded) at angle θ .
- ✓ **Greatest/least** $|z|$: along the line through O and the centre: distance $\pm$ radius.
- ✓ **Greatest/least** $\arg z$: tangents from O ; use right-angled triangles.

Watch out!

$|z - 3 + 2i|$ has centre $3 - 2i$, NOT $-3 + 2i$ or $3 + 2i$. Always rewrite as $|z - (\dots)|$ before reading off the centre.

$|z + 4i|$ is the distance from $-4i$, not from $4i$. Using $(0, 4)$ gives the bisector $x - 2y + 3 = 0$, a different line.

$\arg(z - a) = \theta$ is a HALF-line, not a full line. The opposite direction from a has argument $\theta - \pi$ (or $\theta + \pi$), so drawing the whole line through a loses the mark.