

Differential equations

A-Level Maths 9709 · Topic 3.8 revision notes

Everything from the 2 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Forming differential equations

2. Solving by separating the variables

thelucidstem.com

1 Forming differential equations

Syllabus 3.8

'The rate of change of x ' means $\frac{dx}{dt}$. 'Proportional to $f(x)$ ' means $= kf(x)$ for a constant k . If the quantity is DECREASING, write $\frac{dx}{dt} = -kf(x)$ with $k > 0$, then use the given data to find k .

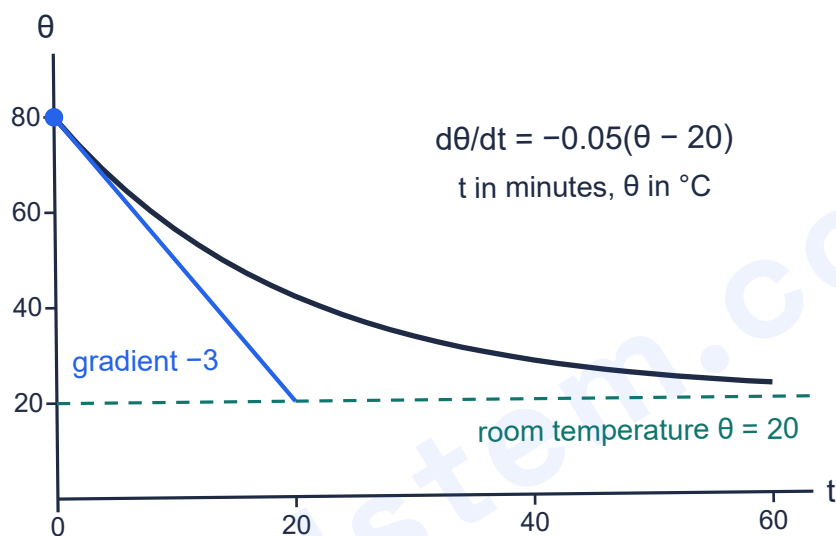


Fig. 1: tea at 80°C cooling in a 20°C room; the rate of cooling is proportional to the excess temperature, so the curve starts with gradient $-0.05 \times 60 = -3$ $^{\circ}\text{C}$ per minute and flattens towards 20°C

Worked example: Newton's law of cooling

A cup of tea is at θ $^{\circ}\text{C}$ at time t minutes, in a room at 20°C . The temperature decreases at a rate proportional to the difference between the temperature of the tea and the room. Write down a differential equation.

1. The quantity changing is θ , with respect to t : $\frac{d\theta}{dt}$.
2. The difference between the tea and the room is $\theta - 20$ (positive while the tea is hotter).
3. The temperature is decreasing, so the derivative is negative: use $-k$ with $k > 0$.

Answer: $\frac{d\theta}{dt} = -k(\theta - 20)$, where k is a positive constant.

Which quantity, with respect to what, depends on what.

- ✓ **Derivative:** rate of change of x means $\frac{dx}{dt}$.
- ✓ **Proportional:** $= kf(x)$; inversely proportional: $= \frac{k}{f(x)}$.
- ✓ **Decreasing:** put in a minus sign and keep $k > 0$.
- ✓ **Constant:** substitute one value and its rate, with the correct sign.
- ✓ **Connected rates:** $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$.

Watch out!

'Rate of change' with no other words means with respect to TIME. Writing $\frac{d\theta}{dx}$ or forgetting the minus sign on a decreasing quantity both lose the first mark, and the error carries into the solution.

If you write $\frac{dh}{dt} = -k\sqrt{h}$, then k must come out POSITIVE. Substituting a rate of $+0.018$ gives $k = -0.02$ and the double negative makes the water rise.

'Growing fastest' means the RATE $\frac{dN}{dt}$ is a maximum. Differentiate the rate with respect to N (or complete the square); do not just set $\frac{dN}{dt} = 0$, which finds where growth STOPS.

2 Solving by separating the variables

Syllabus 3.8

If $\frac{dy}{dx} = f(x)g(y)$, then $\int \frac{1}{g(y)} dy = \int f(x) dx$. Add ONE constant after integrating, use the initial condition to find it, and only then rearrange to make y the subject.

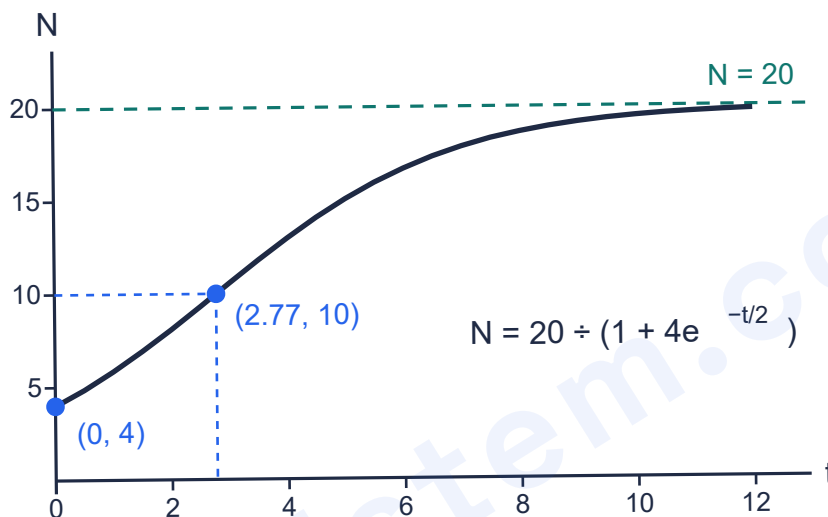


Fig. 1: the solution of $dN/dt = N(20 - N)/40$ with $N = 4$ when $t = 0$; it passes $N = 10$ at $t = 4 \ln 2 = 2.77$ and levels off towards $N = 20$

Worked example: A first separable equation

Solve $\frac{dy}{dx} = \frac{2x}{y}$, given that $y = 3$ when $x = 1$. Find y when $x = 3$.

1. Separate: $y dy = 2x dx$, so $\int y dy = \int 2x dx$.
2. Integrate: $\frac{1}{2}y^2 = x^2 + C$.
3. Use $x = 1$, $y = 3$: $\frac{9}{2} = 1 + C$, so $C = \frac{7}{2}$.
4. Rearrange: $y^2 = 2x^2 + 7$. Since $y = 3 > 0$ at the start, take the positive root:
 $y = \sqrt{2x^2 + 7}$.
5. At $x = 3$: $y = \sqrt{18 + 7} = 5$.

Answer: $y^2 = 2x^2 + 7$; $y = 5$ when $x = 3$.

Separate, integrate, constant, rearrange.

- ✓ **Separate:** $\int \frac{1}{g(y)} dy = \int f(x) dx$; split e^{a+b} into $e^a e^b$ first.
- ✓ **Integrate** both sides, using partial fractions or standard integrals as needed.
- ✓ **One constant**, found from the initial condition straight away.
- ✓ **Rearrange** to the form asked for; exponentiating turns $+C$ into a multiplier.
- ✓ **Interpret:** answer in context with units, and say what happens as $t \rightarrow \infty$.

Watch out!

$\ln y = 2x + C$ gives $y = e^{2x+C} = Ae^{2x}$, NOT $y = e^{2x} + C$. The constant becomes a multiplier when you exponentiate, so find C BEFORE removing the logarithm, or write $A = e^C$.

Do not drop the 20: $\theta = 80e^{-0.05t}$ solves a different equation, and it cools to zero instead of to room temperature. Keep $\theta - 20$ together throughout.

$\int \frac{1}{20 - N} dN = -\ln(20 - N)$, not $+\ln(20 - N)$; the inside has derivative -1 . Losing this sign turns the logistic curve into nonsense.