

Differentiation

A-Level Maths 9709 · Topic 1.7 revision notes

Everything from the 4 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. The gradient of a curve
2. Differentiating powers and the chain rule
3. Tangents, normals and rates of change
4. Stationary points and their nature

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1 The gradient of a curve

Syllabus 1.7

The gradient of a curve at a point is the gradient of the tangent there, the limit of the gradients of chords as the second point slides onto the first. For $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$; bring the power down, then reduce the power by one.

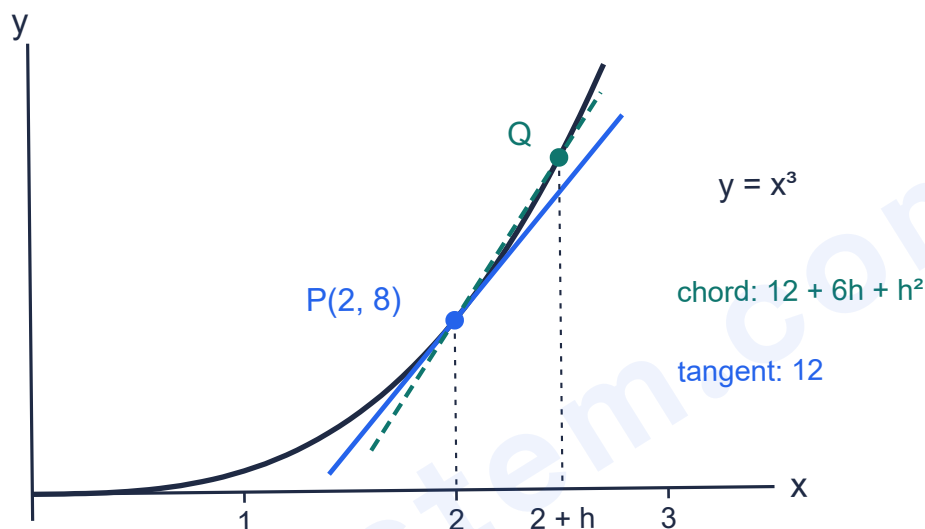


Fig. 1: on $y = x^3$ the chord PQ from $x = 2$ to $x = 2 + h$ has gradient $12 + 6h + h^2$ (15.25 when $h = 0.5$, as drawn); as h shrinks to 0 it tends to 12, the gradient of the tangent at P

Worked example: Chords on a quadratic

The points $P(3, 9)$ and $Q(3 + h, (3 + h)^2)$ lie on $y = x^2$. Find the gradient of the chord PQ in terms of h , and deduce the gradient of the curve at P .

1. Rise: $(3 + h)^2 - 9 = 9 + 6h + h^2 - 9 = 6h + h^2$.
2. Run: $(3 + h) - 3 = h$.
3. Gradient of $PQ = \frac{6h + h^2}{h} = 6 + h$, dividing by h (allowed, since $h \neq 0$ for a chord).
4. Check with numbers: $h = 0.1$ gives 6.1, $h = 0.01$ gives 6.01.
5. As $h \rightarrow 0$, $6 + h \rightarrow 6$.

Answer: Chord gradient $6 + h$; the gradient of the curve at P is 6.

Differentiate, then substitute or solve.

- ✓ **Chord:** $\text{gradient} = \frac{f(a+h) - f(a)}{h}$; simplify, cancel h , then let $h \rightarrow 0$.
- ✓ **Power rule:** $x^n \rightarrow nx^{n-1}$; constants vanish; work term by term.
- ✓ **Gradient at a point:** substitute x into $\frac{dy}{dx}$.
- ✓ **Point with a given gradient:** solve $\frac{dy}{dx} = m$, then use the curve for y .
- ✓ **Second derivative:** differentiate $\frac{dy}{dx}$ again.

Watch out!

Do not put $h = 0$ into $\frac{12h + 6h^2 + h^3}{h}$ straight away: that gives $\frac{0}{0}$, which means nothing. Cancel the h first, THEN let $h \rightarrow 0$.

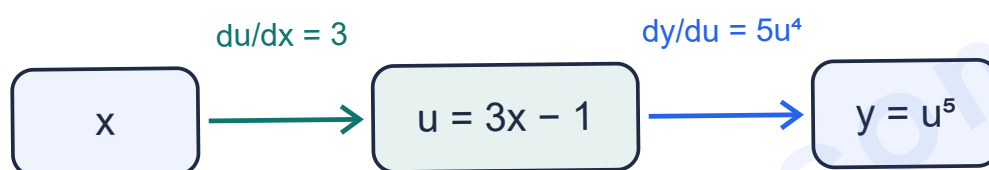
$\frac{d^2y}{dx^2}$ is NOT $\left(\frac{dy}{dx}\right)^2$. For $y = x^3$, $\frac{d^2y}{dx^2} = 6x$, but $\left(\frac{dy}{dx}\right)^2 = 9x^4$.

After solving $\frac{dy}{dx} = 0$ for x , substitute into the equation of the CURVE to get y . Putting x back into $\frac{dy}{dx}$ just gives 0 again.

2 Differentiating powers and the chain rule

Syllabus 1.7

Write every term as kx^n before differentiating. For a function of a function, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$; differentiate the outside leaving the inside alone, then multiply by the derivative of the inside. So $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$.



$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{dy}{du}\right) \times \left(\frac{du}{dx}\right) \\ &= 5u^4 \times 3 = 15(3x - 1)^4\end{aligned}$$

Fig. 1: $y = (3x - 1)^5$ is a chain of two simple functions; multiply the two rates of change to get dy/dx

Worked example: Roots and reciprocals

Given $y = 6\sqrt{x} + \frac{8}{x^2}$, find $\frac{dy}{dx}$ and the gradient of the curve at $x = 4$.

1. Rewrite as powers: $y = 6x^{\frac{1}{2}} + 8x^{-2}$.
2. Differentiate: $6 \times \frac{1}{2}x^{-\frac{1}{2}} = 3x^{-\frac{1}{2}}$ and $8 \times (-2)x^{-3} = -16x^{-3}$.
3. So $\frac{dy}{dx} = \frac{3}{\sqrt{x}} - \frac{16}{x^3}$.
4. At $x = 4$: $\frac{3}{2} - \frac{16}{64} = 1.5 - 0.25 = 1.25$.

Answer: $\frac{dy}{dx} = \frac{3}{\sqrt{x}} - \frac{16}{x^3}$; gradient $\frac{5}{4}$ at $x = 4$.

Rewrite as powers, then differentiate.

- ✓ **Rewrite:** $\sqrt[n]{x^m} = x^{\frac{m}{n}}$ and $\frac{k}{x^n} = kx^{-n}$.
- ✓ **Fractions over x^p :** divide each term, never differentiate top and bottom separately.
- ✓ **Chain rule:** $\frac{d}{dx} [f(x)]^n = n[f(x)]^{n-1} \times f'(x)$.
- ✓ **Linear inside:** $\frac{d}{dx} (ax + b)^n = an(ax + b)^{n-1}$.
- ✓ **Tidy:** write negative and fractional powers back as fractions and roots if asked.

Watch out!

$\frac{1}{2x^2}$ is $\frac{1}{2}x^{-2}$, NOT $2x^{-2}$: only the x^2 moves up with a negative power; the 2 stays in the denominator as $\frac{1}{2}$.

The most common lost mark: forgetting to multiply by the derivative of the inside.

$\frac{d}{dx}(3x - 1)^5$ is $15(3x - 1)^4$, not $5(3x - 1)^4$. Watch the sign too when the inside is something like $4 - x^2$.

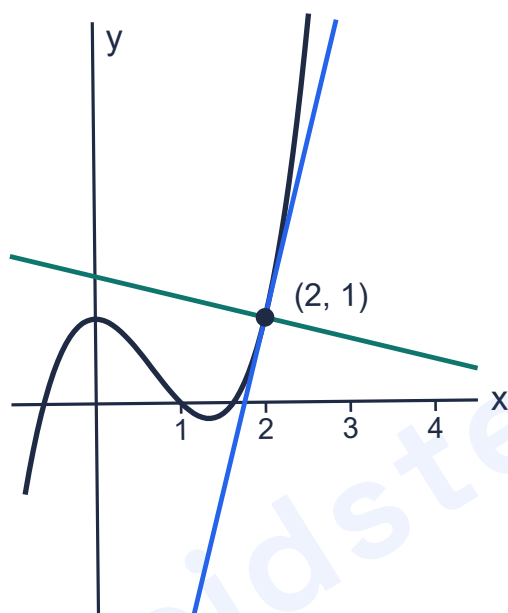
Do not expand $(2x - 3)^4$ before differentiating unless you have to. It is slow, and a single slip in the binomial expansion costs every later mark.

3 Tangents, normals and rates of change

Syllabus 1.7

At a point where the gradient is m , the tangent has gradient m and the normal has gradient $-\frac{1}{m}$. f is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$.

Connected rates: $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$.



$$y = x^3 - 2x^2 + 1$$

$$\text{tangent: } y = 4x - 7$$

$$\text{normal: } x + 4y = 6$$

Fig. 1: the tangent (gradient 4) and the normal (gradient $-\frac{1}{4}$) to $y = x^3 - 2x^2 + 1$ at $(2, 1)$, drawn to equal scales so the right angle is true

Worked example: Tangent and normal to a cubic

Find the equations of the tangent and the normal to $y = x^3 - 2x^2 + 1$ at the point where $x = 2$.

1. Point: $y = 8 - 8 + 1 = 1$, so $(2, 1)$.

2. Gradient: $\frac{dy}{dx} = 3x^2 - 4x = 12 - 8 = 4$.

3. Tangent: $y - 1 = 4(x - 2)$, so $y = 4x - 7$.

4. Normal gradient $= -\frac{1}{4}$: $y - 1 = -\frac{1}{4}(x - 2)$.

5. Multiply by 4: $4y - 4 = -x + 2$, so $x + 4y = 6$.

Answer: Tangent $y = 4x - 7$; normal $x + 4y = 6$.

Point, gradient, line; or know, want, link.

- ✓ **Point:** find both coordinates from the curve.
- ✓ **Gradient:** substitute x into $\frac{dy}{dx}$; normal uses $-\frac{1}{m}$.
- ✓ **Line:** $y - y_1 = m(x - x_1)$; give it in the form asked.
- ✓ **Increasing/decreasing:** factorise $f'(x)$, sketch it, read off where it is positive or negative.
- ✓ **Connected rates:** know one rate, want another, link with a formula, differentiate, then multiply or divide.

Watch out!

The normal gradient is $-\frac{1}{m}$: flip m AND change sign. With $m = 4$, -4 and $\frac{1}{4}$ are both wrong. And never use $\frac{dy}{dx}$ as a formula in the line equation: substitute to get a number first.

For a positive quadratic $f'(x)$, the negative part is between the roots, not outside them. Sketch the parabola of $f'(x)$ before writing the inequality.

Differentiate the formula BEFORE you substitute the particular value. Putting $r = 5$ into $V = \frac{4}{3}\pi r^3$ first turns it into a constant, whose derivative is 0.

4 Stationary points and their nature

Syllabus 1.7

Stationary points are where $\frac{dy}{dx} = 0$. If $\frac{d^2y}{dx^2} < 0$ there, the point is a maximum; if $\frac{d^2y}{dx^2} > 0$, a minimum. If $\frac{d^2y}{dx^2} = 0$, check the sign of $\frac{dy}{dx}$ on either side instead.

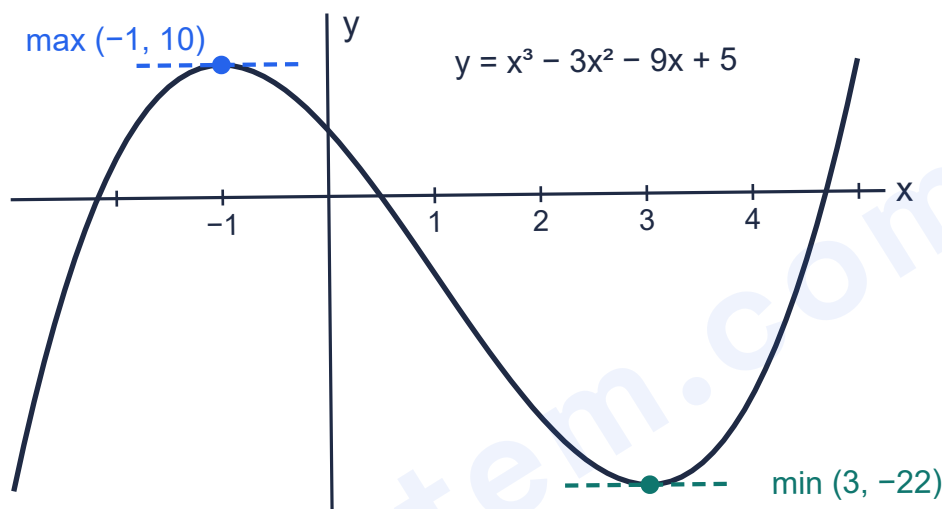


Fig. 1: $y = x^3 - 3x^2 - 9x + 5$ has a maximum at $(-1, 10)$ and a minimum at $(3, -22)$; the tangent is horizontal at both

Worked example: Locate and classify

Find the stationary points of $y = x^3 - 3x^2 - 9x + 5$ and determine their nature.

1. $\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$.
2. $\frac{dy}{dx} = 0$ gives $x = -1$ or $x = 3$.
3. Coordinates from the curve: $x = -1$: $y = -1 - 3 + 9 + 5 = 10$; $x = 3$: $y = 27 - 27 - 27 + 5 = -22$.
4. $\frac{d^2y}{dx^2} = 6x - 6$: at $x = -1$ it is $-12 < 0$, a maximum; at $x = 3$ it is $12 > 0$, a minimum.

Answer: Maximum $(-1, 10)$; minimum $(3, -22)$ (Fig. 1).

Gradient zero, then test the nature.

- ✓ **Find:** solve $\frac{dy}{dx} = 0$; get y from the curve.
- ✓ **Second derivative:** negative means maximum, positive means minimum.
- ✓ **If it is zero:** check the sign of $\frac{dy}{dx}$ just either side.
- ✓ **Sketch:** stationary points, axis crossings, and the behaviour for large $|x|$.
- ✓ **Optimise:** one-variable formula, differentiate, solve, justify max or min, answer the question asked.

Watch out!

A question that says 'find the stationary points' wants both coordinates. Leaving the answer as $x = -1$ and $x = 3$ loses the accuracy marks.

$\frac{d^2y}{dx^2} = 0$ does NOT tell you the nature. Go back to the sign of $\frac{dy}{dx}$ either side of the point, choosing x -values close to it.

Answer the question asked: 'find the maximum volume' wants $V = 128$, not $x = 2$. And always show that the point IS a maximum, with the second derivative or a sign check.