

Discrete random variables

A-Level Maths 9709 · Topic 5.4 revision notes

Everything from the 3 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Discrete probability distributions, $E(X)$ and $\text{Var}(X)$

2. The binomial distribution

3. The geometric distribution

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1 Discrete probability distributions, $E(X)$ and $\text{Var}(X)$

Syllabus 5.4

The probabilities in a distribution add up to 1. $E(X) = \sum x p$ is the long-run mean, and $\text{Var}(X) = \sum x^2 p - [E(X)]^2$ measures the spread: the same 'mean of the squares minus the square of the mean' as for data.

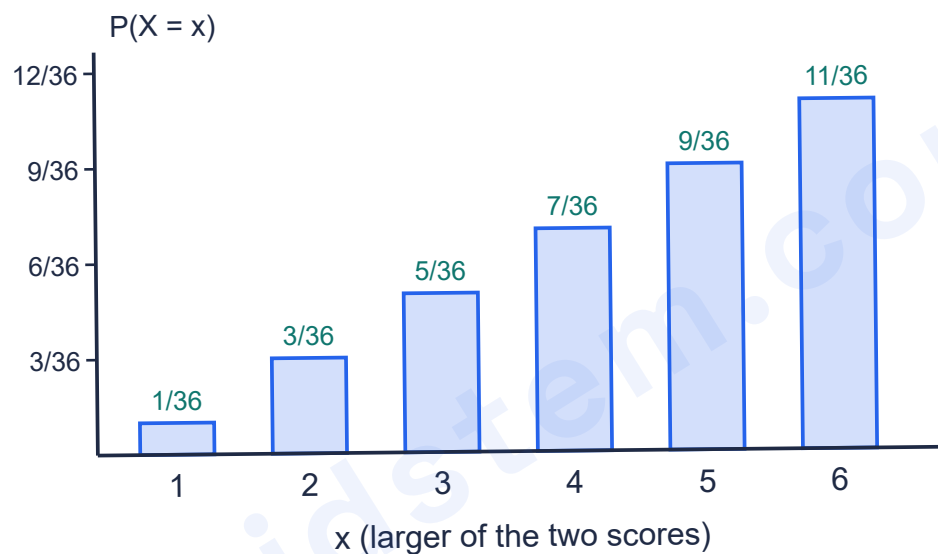


Fig. 1: X is the larger score when two fair dice are thrown. $P(X = x) = (2x - 1)/36$, and the six probabilities add up to $36/36 = 1$

Worked example: Finding a constant

The random variable X has $P(X = x) = kx$ for $x = 1, 2, 3, 4$. Find k and $P(X \geq 3)$.

1. $k(1) + k(2) + k(3) + k(4) = 1$, so $10k = 1$ and $k = 0.1$.
2. $P(X \geq 3) = P(X = 3) + P(X = 4) = 0.3 + 0.4$.

Answer: $k = 0.1$, $P(X \geq 3) = 0.7$

Values, probabilities, then sums.

- ✓ **Values:** list every value X can take.
- ✓ **Probabilities:** trees, sample spaces or combinations; check they add to 1.
- ✓ **Expectation:** $E(X) = \sum xp$.
- ✓ **Variance:** $E(X^2) - [E(X)]^2$ with $E(X^2) = \sum x^2p$.
- ✓ **Unknowns:** one equation from $\sum p = 1$, another from the given $E(X)$.

Watch out!

Forgetting that $P(X \geq 3)$ includes 3. 'At least 3' and 'more than 2' are the same event for whole-number values; 'more than 3' is not.

Missing a route in $P(X = 1)$. One green can come first OR second; red-green and green-red are both needed.

Stopping at $E(X^2)$, or writing $\text{Var}(X) = E(X^2) - E(X)$. The expectation must be SQUARED before subtracting.

Using the winnings instead of the profit. Expected winnings are $\frac{7}{6} = \$1.17$; you must still subtract the \$1.50 stake.

2 The binomial distribution

Syllabus 5.4

If $X \sim B(n, p)$, then $P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$ for $r = 0, 1, \dots, n$, with $E(X) = np$ and $\text{Var}(X) = np(1 - p)$.

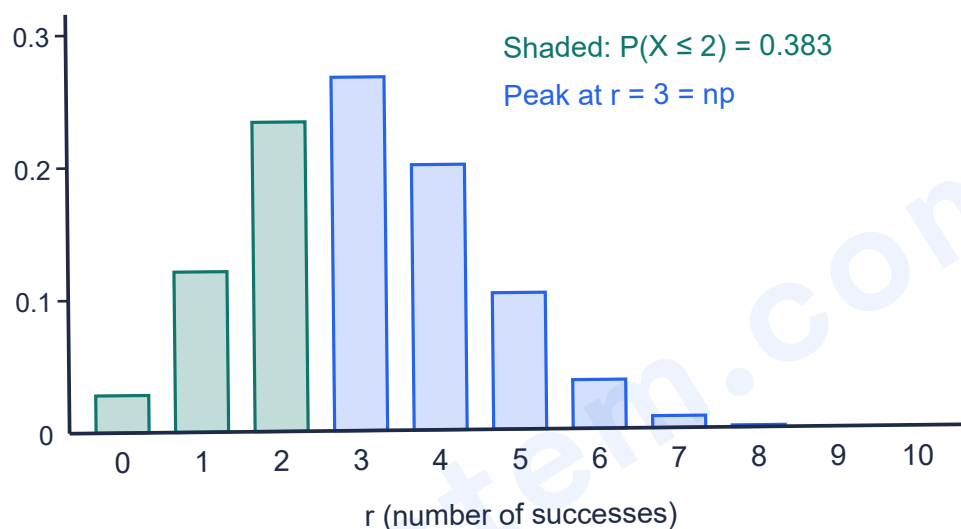


Fig. 1: $X \sim B(10, 0.3)$ drawn to scale. The three shaded bars add to $P(X \leq 2) = 0.0282 + 0.1211 + 0.2335 = 0.383$

Worked example: One exact probability

$X \sim B(10, 0.3)$. Find $P(X = 3)$.

1. $\binom{10}{3} = 120$ ways to place the 3 successes among 10 trials.
2. Each such sequence has probability $0.3^3 \times 0.7^7$.
3. $120 \times 0.027 \times 0.0823543 = 0.26683$.

Answer: 0.267 (3 s.f.)

Define X , then $B(n, p)$.

- ✓ **Model:** state X , n and p , and check the four conditions.
- ✓ **Exact:** $P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$.
- ✓ **Cumulative:** turn the words into an inequality; use the complement when shorter.
- ✓ **Mean and variance:** np and $np(1 - p)$.
- ✓ **Least n:** set up the inequality and try values either side.

Watch out!

Leaving out $\binom{n}{r}$. $0.3^3 \times 0.7^7$

is the probability of ONE particular order, such as SSSFFFFFFF; there are 120 orders.

Treating 'more than 2' as $X \geq 2$.
For whole numbers 'more than 2' starts at 3.

Stopping at $n = 20$ because 0.0115 'is about 0.01'. The inequality must actually hold, so check the value either side.

Rounding the first answer to 0.74 before using it again. Keep at least 4 significant figures in intermediate values so the final 3 s.f. is right.

3 The geometric distribution

Syllabus 5.4

If $X \sim \text{Geo}(p)$, X is the number of the trial on which the first success occurs: $P(X = r) = p(1 - p)^{r-1}$ for $r = 1, 2, 3, \dots$. Also $P(X > r) = (1 - p)^r$ (the first r trials all fail) and $E(X) = \frac{1}{p}$.

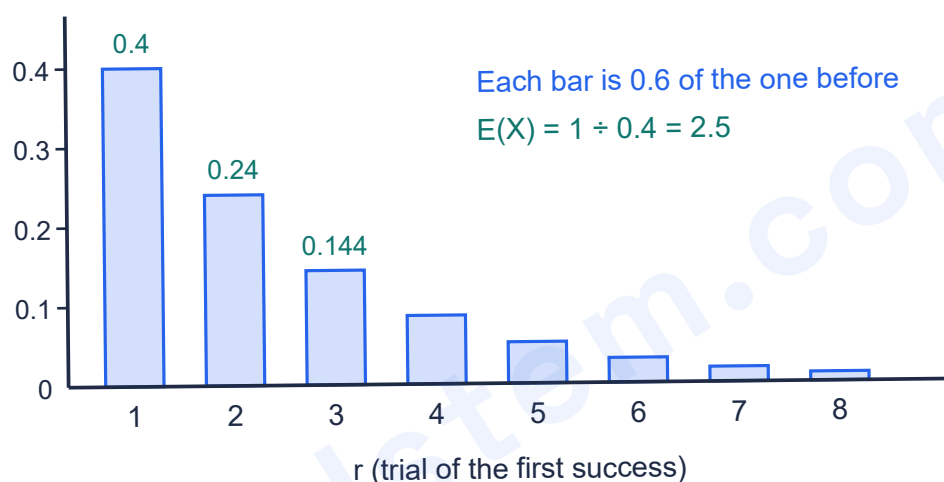


Fig. 1: $X \sim \text{Geo}(0.4)$ drawn to scale for $r = 1$ to 8. The most likely value is always $r = 1$, and the probabilities shrink by the factor $1 - p$ each time

Worked example: An exact probability

A fair die is thrown until a six appears. Find the probability that the first six occurs on the 4th throw.

1. $X \sim \text{Geo}\left(\frac{1}{6}\right)$.
2. Three non-sixes then a six: $\left(\frac{5}{6}\right)^3 \times \frac{1}{6}$.
3. $= \frac{125}{1296}$.

Answer: $\frac{125}{1296} = 0.0965$ (3 s.f.)

Fail $r - 1$ times, then succeed.

- ✓ **Model:** independent trials, constant p , counting up to the first success: $X \sim \text{Geo}(p)$.
- ✓ **Exact:** $P(X = r) = p q^{r-1}$.
- ✓ **Cumulative:** $P(X > r) = q^r$,
 $P(X \leq r) = 1 - q^r$.
- ✓ **Mean:** $E(X) = \frac{1}{p}$.
- ✓ **Least n :** solve $q^n < \dots$ by trying values, to 4 s.f.

Watch out!

Using $(1 - p)^r p$. If the first success is on trial r , only $r - 1$ trials fail before it, so the power is $r - 1$.

Mixing up $X > r$ and $X \geq r$.
 $P(X \geq 4) = q^3$, not q^4 : only the first three trials have to fail.

Taking $n = 8$ because 0.75^8 'rounds to 0.1'. The inequality is strict, and 0.1001 is not less than 0.1. Work to 4 significant figures.