

# Functions

## A-Level Maths 9709 · Topic 1.2 revision notes

Everything from the 3 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Functions: domain, range and composition
2. Inverse functions
3. Transformations of graphs

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# 1 Functions: domain, range and composition

## Syllabus 1.2

The domain is the set of inputs; the range is the set of outputs it produces.  $fg(x) = f(g(x))$ : apply  $g$  FIRST. The composite  $fg$  exists only when the range of  $g$  lies within the domain of  $f$ .

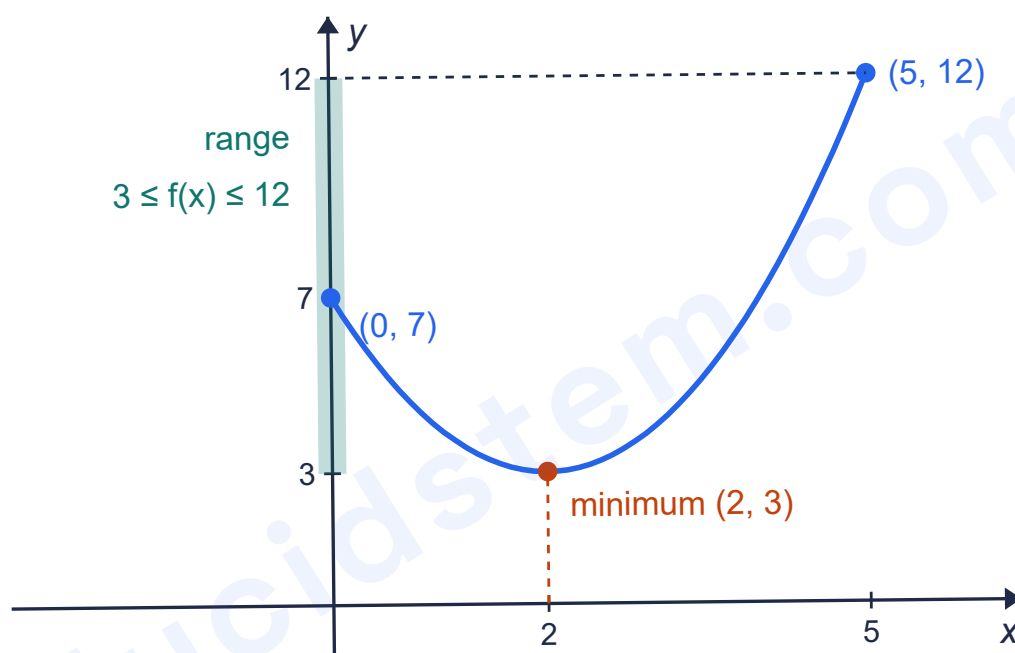


Fig. 1:  $f(x) = (x - 2)^2 + 3$  for  $0 \leq x \leq 5$ . The least output is at the vertex, not at an end, so the range is  $3 \leq f(x) \leq 12$

### Worked example: Linear and reciprocal

Find the range of (a)  $f(x) = 2x - 3$  for  $-1 \leq x \leq 4$  (b)  $g : x \mapsto \frac{1}{x}$  for  $x \geq 1$ .

1. (a)  $f$  is increasing, so the ends give the extremes:  $f(-1) = -5$  and  $f(4) = 5$ .

2. (b)  $g(1) = 1$ . As  $x$  increases,  $\frac{1}{x}$  decreases towards 0 but never reaches it.

**Answer:** (a)  $-5 \leq f(x) \leq 5$  (b)  $0 < g(x) \leq 1$

### Sketch for the range, inside-out for composites.

- ✓ **Range:** sketch over the domain; check both ends and any vertex inside it.
- ✓ **Write** the range in terms of  $f(x)$ , with  $<$  or  $\leq$  chosen carefully.
- ✓ **One-one:** no horizontal line meets the graph twice; restrict to one side of a vertex.
- ✓ **Composite:**  $fg(x) = f(g(x))$ , so  $g$  acts first.
- ✓ **Existence:**  $fg$  needs the range of  $g$  inside the domain of  $f$ .

### Watch out!

Do not just substitute the two ends of the domain. For Fig. 1 that gives  $7 \leq f(x) \leq 12$ , missing every output between 3 and 7. Always look for a vertex inside the domain.

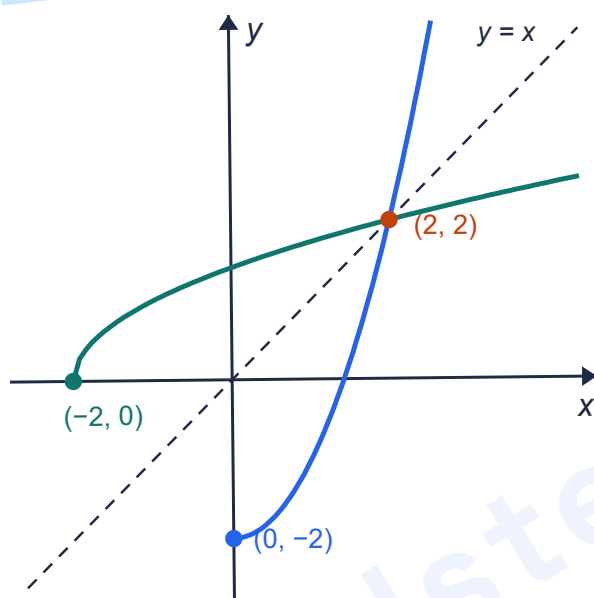
The least  $k$  is the  $x$ -coordinate of the vertex, not its  $y$ -coordinate. For  $(x - 3)^2 - 8$  the answer is  $k = 3$ , not  $-8$ .

$fg(x)$  is NOT  $f(x) \times g(x)$ . With  $f(x) = 2x + 1$  and  $g(x) = x^2$ , the product is  $2x^3 + x^2$ , but  $fg(x) = 2x^2 + 1$ .

## 2 Inverse functions

### Syllabus 1.2

To find  $f^{-1}$ : write  $y = f(x)$ , make  $x$  the subject, then swap the letters. The domain of  $f^{-1}$  is the range of  $f$ , the range of  $f^{-1}$  is the domain of  $f$ , and the graph of  $y = f^{-1}(x)$  is the reflection of  $y = f(x)$  in the line  $y = x$ .



$$y = f(x) = x^2 - 2, x \geq 0$$

$$y = f^{-1}(x) = \sqrt{x+2}, x \geq -2$$

mirror line  $y = x$

they meet on  $y = x$  at  $(2, 2)$

Fig. 1:  $f(x) = x^2 - 2$  for  $x \geq 0$  and its inverse  $f^{-1}(x) = \sqrt{x+2}$  are mirror images in  $y = x$ ; they meet on that line at  $(2, 2)$

### Worked example: Linear

Find  $f^{-1}(x)$  for  $f(x) = 3x - 5$ ,  $x \in \mathbb{R}$ .

1. Write  $y = 3x - 5$ .

2. Make  $x$  the subject:  $3x = y + 5 \Rightarrow x = \frac{y+5}{3}$ .

3. Swap letters to write the inverse as a function of  $x$ .

**Answer:**  $f^{-1}(x) = \frac{x+5}{3}$ , for  $x \in \mathbb{R}$

*Make  $x$  the subject, then swap.*

- ✓ **Check** the function is one-one on its domain.
- ✓ **Write**  $y = f(x)$  and make  $x$  the subject (complete the square for a quadratic).
- ✓ **Choose** the sign of any square root to match the domain of  $f$ .
- ✓ **Swap** letters to write  $f^{-1}(x)$ .
- ✓ **State** the domain of  $f^{-1}$ : it is the range of  $f$ .

**Watch out!**

$f^{-1}(x)$  is NOT  $\frac{1}{f(x)}$ . For

$f(x) = 3x - 5$ ,  $\frac{1}{3x - 5}$  is a reciprocal; the inverse is  $\frac{x + 5}{3}$ .

Writing  $\pm\sqrt{\quad}$  in the final answer is wrong: an inverse function gives ONE output. Test your choice:  $g^{-1}$  must output values  $\leq \frac{1}{3}$ , and  $\frac{1 - \sqrt{x+5}}{3}$  does.

Sketch  $f^{-1}$  only over the RANGE of  $f$ . In Fig. 1 the inverse starts at  $(-2, 0)$ ; drawing  $\sqrt{x+2}$  to the left of  $-2$ , or reflecting the whole parabola, earns no marks.

### 3 Transformations of graphs

#### Syllabus 1.2

Changes *OUTSIDE* the function act on  $y$  as you would expect:  $f(x) + a$  moves up  $a$ ,  $af(x)$  stretches by factor  $a$  parallel to the  $y$ -axis. Changes *INSIDE* act on  $x$  the opposite way:  $f(x + a)$  moves *LEFT*  $a$ ,  $f(ax)$  stretches by factor  $\frac{1}{a}$  parallel to the  $x$ -axis.

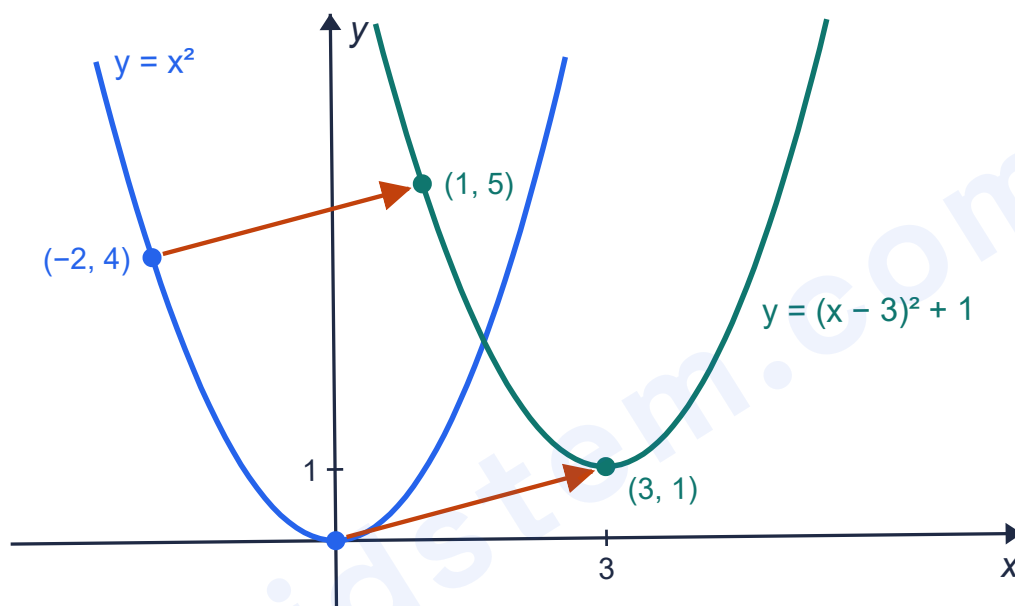


Fig. 1:  $y = (x - 3)^2 + 1$  is  $y = x^2$  translated 3 units right and 1 unit up; every point, such as  $(0, 0)$  and  $(-2, 4)$ , moves by the same vector

#### Worked example: Describe a translation

Describe the transformation that maps  $y = x^2$  onto  $y = (x - 3)^2 + 1$  (Fig. 1).

1.  $x$  is replaced by  $x - 3$ : the graph moves 3 units in the positive  $x$ -direction.
2.  $+1$  is added outside: the graph moves 1 unit up.
3. Check with the vertex:  $(0, 0)$  moves to  $(3, 1)$ , the vertex of the new curve.

**Answer:** A translation by  $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

*Outside acts on  $y$ ; inside acts on  $x$ , backwards.*

- ✓ **Translation**  $\begin{pmatrix} p \\ q \end{pmatrix}$ :  $y = f(x - p) + q$
- ✓ **Reflection**:  $-f(x)$  in the  $x$ -axis;  $f(-x)$  in the  $y$ -axis.
- ✓ **Stretch**:  $af(x)$  parallel to the  $y$ -axis, factor  $a$ ;  $f(ax)$  parallel to the  $x$ -axis, factor  $\frac{1}{a}$ .
- ✓ **Describe fully**: name the transformation, the axis or direction, and the vector or factor.
- ✓ **Combine** one step at a time, rewriting the equation after each, to check the order.

### *Watch out!*

$y = f(x + 4)$  moves the graph 4 units LEFT, not right. A point  $(2, 5)$  on  $y = f(x)$  goes to  $(-2, 5)$ : you need  $x + 4 = 2$  to get the same output.

$y = f(2x)$  is a stretch by factor  $\frac{1}{2}$ , not 2: the graph is squeezed towards the  $y$ -axis. And always say 'parallel to the  $x$ -axis' or 'parallel to the  $y$ -axis'; 'a stretch of factor 2' alone scores nothing.

Inside the bracket, translations and stretches interact:  $f(2x - 6)$  is NOT 'translate 6 right, stretch factor  $\frac{1}{2}$ ' unless you do them in that order. Test any sequence by building the equation.