

Differentiation

A-Level Maths 9709 · Topic 3.4 revision notes

Everything from the 3 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Derivatives of exponentials, logarithms and trigonometric functions
2. The product and quotient rules
3. Parametric and implicit differentiation

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1 Derivatives of exponentials, logarithms and trigonometric functions

Syllabus 3.4

$\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}\ln x = \frac{1}{x}$, $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, $\frac{d}{dx}\tan x = \sec^2 x$, $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$ (x in radians). For a composite, multiply by the derivative of the inside: $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$ and $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$.

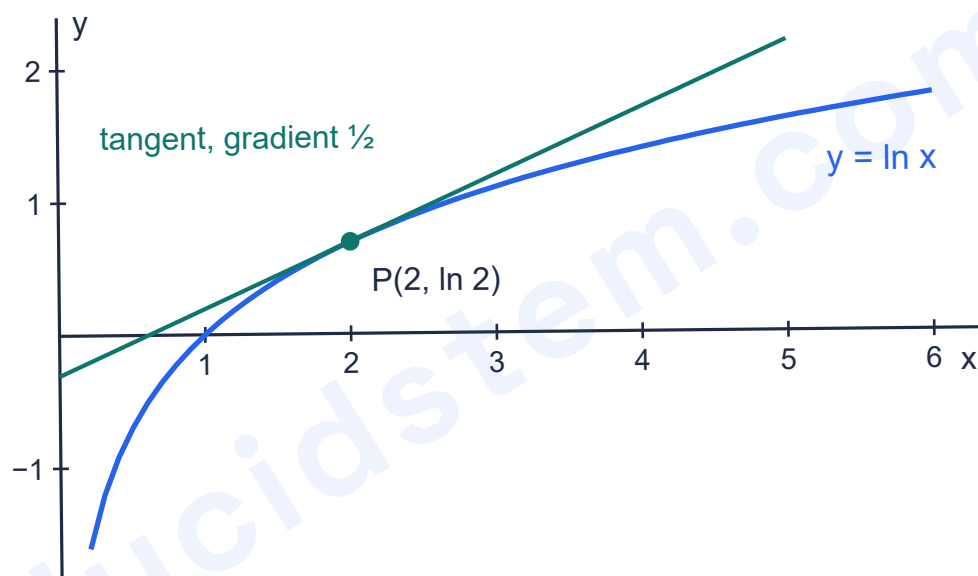


Fig. 1: the gradient of $y = \ln x$ is $1/x$. At $P(2, \ln 2)$ the tangent has gradient $\frac{1}{2}$, and it crosses the y-axis at $\ln 2 - 1 \approx -0.307$. As x grows, $1/x$ shrinks, so the curve keeps rising but ever more slowly

Worked example: Exponentials

Differentiate $y = 5e^{3x} - 2e^{-x} + e^{1-x^2}$.

1. $\frac{d}{dx}5e^{3x} = 5 \times 3e^{3x} = 15e^{3x}$.
2. $\frac{d}{dx}(-2e^{-x}) = -2 \times (-1)e^{-x} = 2e^{-x}$.
3. $\frac{d}{dx}e^{1-x^2} = -2xe^{1-x^2}$ (derivative of the power is $-2x$).

Answer: $\frac{dy}{dx} = 15e^{3x} + 2e^{-x} - 2xe^{1-x^2}$

Standard result, times the inside derivative.

✓ **Exponential:** $\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}.$

✓ **Logarithm:** $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)};$
simplify with log laws first.

✓ **Trig (radians):** $\sin \rightarrow \cos, \cos \rightarrow -\sin, \tan \rightarrow \sec^2.$

✓ **Paper 3:** $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}.$

✓ **Stationary points:** set $\frac{dy}{dx} = 0$ and use $\ln$ to solve exponential equations exactly.

Watch out!

$\frac{d}{dx} \ln(4x) = \frac{4}{4x} = \frac{1}{x},$ NOT $\frac{4}{x}$
and not $\frac{1}{4x}.$ (Check: $\ln 4x = \ln 4 + \ln x,$ and $\ln 4$ is a constant.)

The minus sign belongs to cosine:

$\frac{d}{dx} \cos x = -\sin x,$ but
 $\frac{d}{dx} \sin x = +\cos x.$ A memory aid: going round $\sin, \cos, -\sin, -\cos,$ each step is one derivative.

$\tan^{-1} x$ is the inverse function,

not $\frac{1}{\tan x} = \cot x.$ Their derivatives are completely

different: $\frac{1}{1+x^2}$ versus
 $-\operatorname{cosec}^2 x.$

When solving $e^{2x} = 3,$ take $\ln$ of both sides: $2x = \ln 3.$ Writing

$2x = e^3$ or $x = \ln \frac{3}{2}$ are the two classic slips.

2 The product and quotient rules

Syllabus 3.4

Product rule: if $y = uv$, then $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$. Quotient rule: if $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$. In the quotient rule the order matters: bottom times derivative of top comes first.

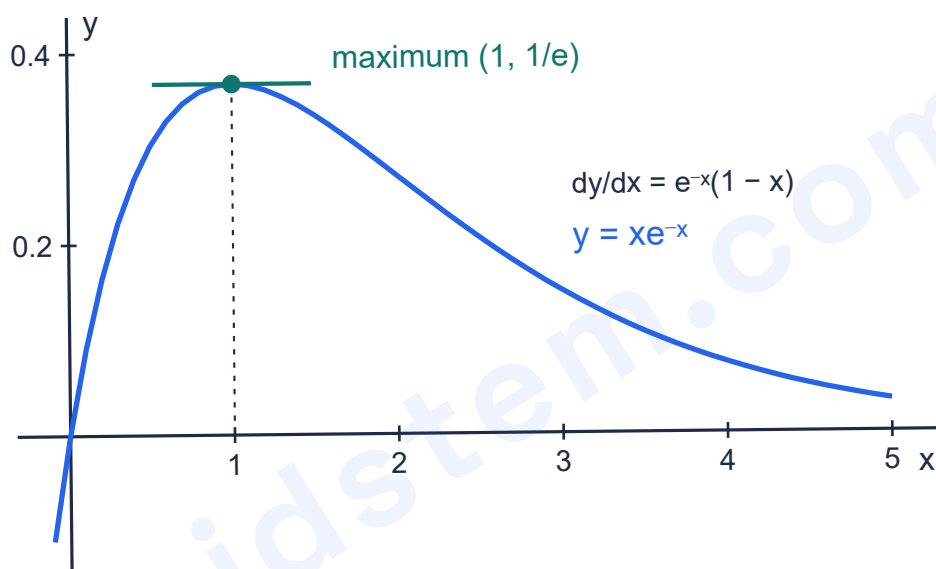


Fig. 1: $y = xe^{-x}$ drawn to scale. The product rule gives $dy/dx = e^{-x}(1-x)$, which is zero only at $x = 1$, so the curve has a horizontal tangent at its maximum point $(1, 1/e) \approx (1, 0.368)$

Worked example: Power times log

Differentiate $y = x^2 \ln x$.

$$1. u = x^2, \frac{du}{dx} = 2x; v = \ln x, \frac{dv}{dx} = \frac{1}{x}.$$

$$2. \frac{dy}{dx} = x^2 \times \frac{1}{x} + \ln x \times 2x.$$

$$3. = x + 2x \ln x = x(1 + 2 \ln x).$$

Answer: $\frac{dy}{dx} = x(1 + 2 \ln x)$

Name u and v , differentiate each, combine.

- ✓ **Product:** $(uv)' = uv' + vu'$.
- ✓ **Quotient:** $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$.
- ✓ **Chain rule** inside u or v (for example e^{3x} , $\sin 2x$).
- ✓ **Factorise** the answer, especially common exponentials.
- ✓ **Stationary points:** a product is zero when a factor is; a fraction is zero when its numerator is.

Watch out!

$\frac{d}{dx}(uv) \neq \frac{du}{dx} \times \frac{dv}{dx}$. For $y = x^2 e^{3x}$, multiplying derivatives gives $6xe^{3x}$, which is wrong; the product rule gives $xe^{3x}(2 + 3x)$.

Swapping the terms in the numerator, $uv' - vu'$, changes the sign of the whole answer. For $\frac{2x - 4}{3x + 2}$ it would give $-\frac{16}{(3x + 2)^2}$. Also, never forget to square the denominator.

When solving $e^{-x}(1 - x) = 0$, do not 'divide by zero' or try to solve $e^{-x} = 0$; it has no solution. Only the other factor gives stationary points.

3 Parametric and implicit differentiation

Syllabus 3.4

Parametric: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Implicit: differentiate every term with respect to x , treating y as a function of x , so $\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx}$ and $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$; then make $\frac{dy}{dx}$ the subject.

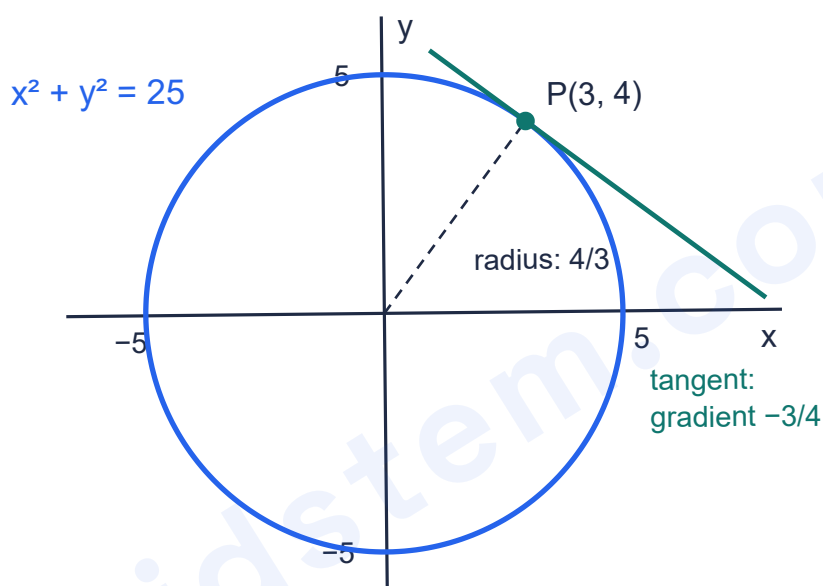


Fig. 1: implicit differentiation of $x^2 + y^2 = 25$ gives $dy/dx = -x/y$, which is $-3/4$ at $P(3, 4)$. The tangent $3x + 4y = 25$ is perpendicular to the radius OP , whose gradient is $4/3$, exactly as circle geometry says

Worked example: A tangent

A curve has parametric equations $x = 3t^2$, $y = 2t^3$. Find the equation of the tangent at the point where $t = 2$.

1. $\frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6t^2$, so $\frac{dy}{dx} = \frac{6t^2}{6t} = t$.
2. At $t = 2$: the point is $(12, 16)$ and the gradient is 2.
3. $y - 16 = 2(x - 12)$.

Answer: $y = 2x - 8$

Differentiate everything, then rearrange.

- ✓ **Parametric:** $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; find t at the point first.
- ✓ **Implicit:** differentiate every term with respect to x ; each y term gains $\frac{dy}{dx}$.
- ✓ **Products** such as xy need the product rule.
- ✓ **Collect** the $\frac{dy}{dx}$ terms on one side and factorise.
- ✓ **Stationary points:** numerator zero, then substitute back into the curve.

Watch out!

The formula is $\frac{dy/dt}{dx/dt}$, y -derivative on TOP. Dividing the other way gives $\frac{dx}{dy}$, the reciprocal of the gradient.

$\frac{d}{dx}(3xy)$ is NOT $3\frac{dy}{dx}$ or $3y$. It is a product: $3x\frac{dy}{dx} + 3y$.

Missing one of the two terms is the most common error in implicit questions.

Do not stop at $y = 2x$. That is a line, not a point; you must substitute into the ORIGINAL curve to find where the line meets it.