

Integration

A-Level Maths 9709 · Topic 3.5 revision notes

Everything from the 6 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

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|---|---------------------------------------|
| 1. <u>Integrating exponentials, $1/(ax + b)$ and trigonometric functions</u> | 4. <u>Integration by substitution</u> |
| 2. <u>Integration using trigonometric identities</u> | 5. <u>Integration by parts</u> |
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1 Integrating exponentials, $1/(ax + b)$ and trigonometric functions

Syllabus 3.5

If $\frac{d}{dx}F(x) = f(x)$, then $\int f(ax + b) dx = \frac{1}{a}F(ax + b) + c$. Integrate as if the bracket were just x , then divide by a , the coefficient of x . Angles are always in radians.

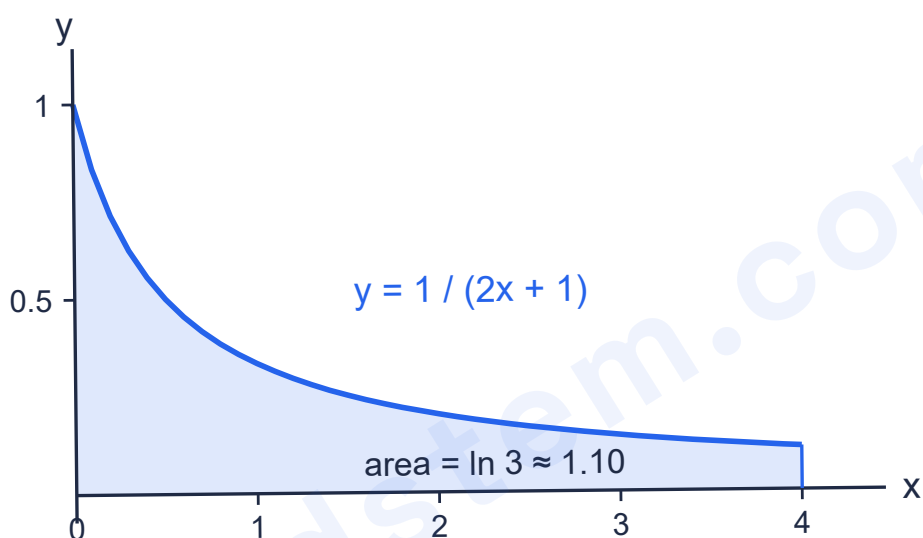


Fig. 1: the area under $y = 1/(2x + 1)$ from $x = 0$ to $x = 4$ is the integral of $1/(2x + 1)$, which is $(1/2) \ln(2x + 1)$; between the limits this gives $(1/2) \ln 9 = \ln 3$, about 1.10 square units

Worked example: Indefinite integral

Find $\int \left(e^{3x-1} + \frac{4}{2x+5} \right) dx$.

1. For e^{3x-1} , $a = 3$: $\int e^{3x-1} dx = \frac{1}{3}e^{3x-1}$.

2. For $\frac{4}{2x+5}$, take the 4 outside and use $a = 2$: $4 \times \frac{1}{2} \ln |2x + 5| = 2 \ln |2x + 5|$.

3. Add one constant of integration for the whole answer.

Answer: $\frac{1}{3}e^{3x-1} + 2 \ln |2x + 5| + c$

Integrate the outside, then divide by a.

- ✓ **Recognise** the standard form: e^{ax+b} , $\frac{1}{ax+b}$, $\sin$, $\cos$, $\sec^2$ or $\frac{1}{x^2+a^2}$.
- ✓ **Divide** by the coefficient of x inside the bracket.
- ✓ **Signs:** $\int \sin = -\cos$, $\int \cos = +\sin$.
- ✓ **Radians** for every trigonometric limit.
- ✓ **Exact answers:** tidy logs with the log laws and keep π , e and surds.

Watch out!

$\int \frac{1}{2x+1} dx$ is NOT $\ln|2x+1|$. Differentiate $\ln|2x+1|$ and you get $\frac{2}{2x+1}$, twice too big, so the answer needs $\frac{1}{2}$ in front.

Equally, never multiply by a : that is differentiating, not integrating.

$\int \sin x dx = -\cos x$, with a MINUS. The most common lost mark is writing $+\cos x$. A quick check: differentiate your answer and see whether you get back exactly what you started with.

Do not forget the $\frac{1}{a}$ in front of $\tan^{-1}$, and do not leave $\tan^{-1} 1$ in degrees: $\tan^{-1} 1 = \frac{\pi}{4}$, never 45, inside a calculus answer.

2 Integration using trigonometric identities

Syllabus 3.5

Squares of sine and cosine are integrated by lowering the power with the double-angle formula: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$. For $\tan^2 x$ use $\tan^2 x = \sec^2 x - 1$.

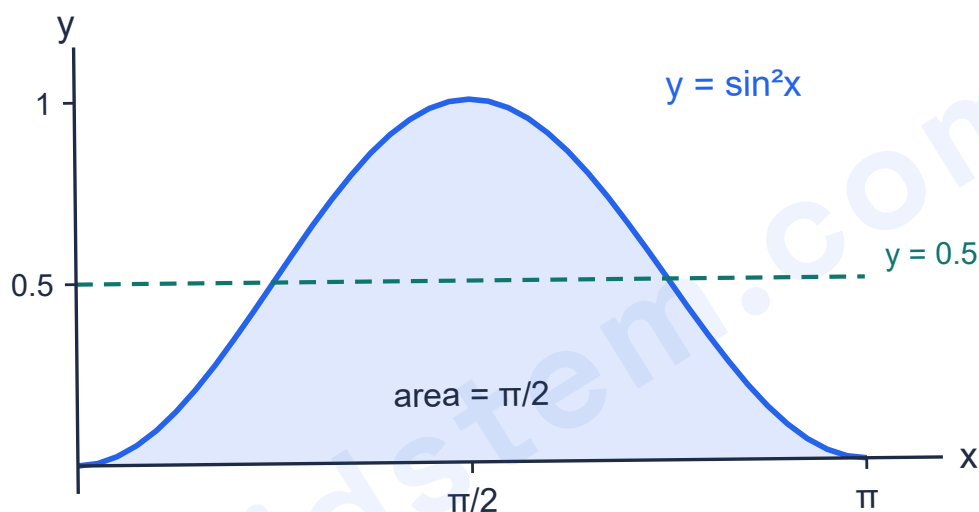


Fig. 1: $y = \sin^2 x = 1/2 - (1/2)\cos 2x$ oscillates about the line $y = 1/2$; over 0 to π the cosine part integrates to zero, so the area is $(1/2) \times \pi = \pi/2$

Worked example: Integrate sine squared

Find $\int \sin^2 x \, dx$.

1. Use $\cos 2x = 1 - 2\sin^2 x$, so $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

2. $\int \frac{1}{2} dx = \frac{x}{2}$.

3. $\int \frac{1}{2} \cos 2x \, dx = \frac{1}{2} \times \frac{1}{2} \sin 2x = \frac{1}{4} \sin 2x$, and it is subtracted.

Answer: $\frac{x}{2} - \frac{1}{4} \sin 2x + c$

Choose the identity, then integrate term by term.

- ✓ **Squares of sin or cos:** use $\cos 2\theta$ to lower the power; the angle doubles.
- ✓ **$\tan^2$:** replace by $\sec^2 - 1$.
- ✓ **Products:** $\sin x \cos x = \frac{1}{2} \sin 2x$.
- ✓ **Brackets:** expand, then use $\sin^2 + \cos^2 = 1$.
- ✓ **Evaluate** in radians and keep π and surds exact.

Watch out!

$\int \sin^2 x \, dx$ is NOT $\frac{\sin^3 x}{3}$. The power rule only works on x^n ;

differentiating $\frac{\sin^3 x}{3}$ gives $\sin^2 x \cos x$, not $\sin^2 x$. And when the angle is $2x$, the identity brings in $\cos 4x$, not $\cos 2x$.

$\tan^2 x = \sec^2 x - 1$, not $\sec^2 x + 1$. Check with $x = 0$: $\tan^2 0 = 0$ and $\sec^2 0 - 1 = 0$. The wrong sign turns $1 - \frac{\pi}{4}$ into $1 + \frac{\pi}{4}$.

When you square $\frac{1 + \cos 2x}{2}$, square the 2 in the denominator as well: the factor in front is $\frac{1}{4}$, not $\frac{1}{2}$.

3 Integration using partial fractions and $f'(x)/f(x)$

Syllabus 3.5

$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$. For any other proper rational function, write it in partial fractions; then $\frac{A}{ax+b} \rightarrow \frac{A}{a} \ln |ax+b|$, $\frac{B}{(ax+b)^2} \rightarrow -\frac{B}{a(ax+b)}$, and $\frac{Cx+D}{x^2+k^2}$ splits into a log and an inverse tan.

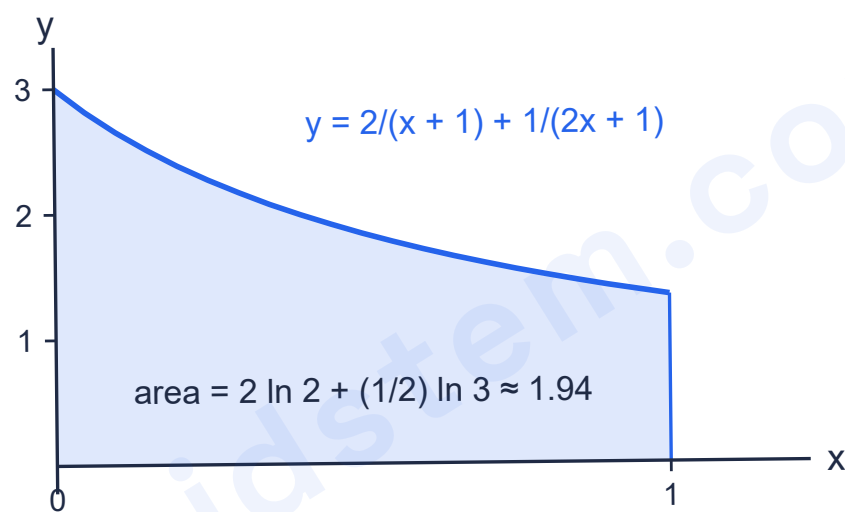


Fig. 1: $y = (5x+3)/((x+1)(2x+1))$ split into partial fractions as $2/(x+1) + 1/(2x+1)$; each piece integrates to a logarithm, so the shaded area from $x=0$ to $x=1$ is $2 \ln 2 + (1/2) \ln 3$, about 1.94

Worked example: Fix the constant

Find the exact value of $\int_0^2 \frac{x}{x^2+4} dx$.

1. The derivative of x^2+4 is $2x$. The top is $x = \frac{1}{2}(2x)$.
2. So $\int \frac{x}{x^2+4} dx = \frac{1}{2} \ln(x^2+4)$ (no modulus: $x^2+4 > 0$).
3. Limits: $\frac{1}{2} \ln 8 - \frac{1}{2} \ln 4 = \frac{1}{2} \ln 2$.

Answer: $\frac{1}{2} \ln 2$

Is the top the derivative of the bottom?

- ✓ **Check first:** if the numerator is $kf'(x)$, the answer is $k \ln |f(x)|$.
- ✓ **Improper?** If the top has the same degree as the bottom, start with a constant term.
- ✓ **Split** into partial fractions; substitute roots, then compare coefficients.
- ✓ **Integrate** each piece: $\ln$ for linear, power rule for squared, $\ln$ plus $\tan^{-1}$ for $x^2 + k^2$.
- ✓ **Tidy** with the log laws; give exact values unless asked otherwise.

Watch out!

$$\int \frac{x}{x^2 + 4} dx \text{ is not } \ln(x^2 + 4)$$

: that differentiates to $\frac{2x}{x^2 + 4}$.

Always differentiate your log mentally and compare the numerator, then fix the constant in front.

$$\frac{1}{2x + 1} \text{ integrates to } \frac{1}{2} \ln |2x + 1|, \text{ not } \ln |2x + 1|.$$

Partial fractions with a coefficient of x other than 1 are where this mark is most often lost.

$\frac{1}{(x + 1)^2}$ does NOT integrate to a logarithm. Write it as $(x + 1)^{-2}$ and use the power rule:

$$-\frac{1}{x + 1}. \text{ Likewise } \frac{1}{x^2 + 1} \text{ gives } \tan^{-1} x, \text{ not } \ln(x^2 + 1).$$

4 Integration by substitution

Syllabus 3.5

With a substitution $u = g(x)$, replace three things: the integrand (in terms of u), dx (using $\frac{du}{dx}$), and, for a definite integral, the limits (x -values become u -values). Then the whole integral is in u and no x may remain.

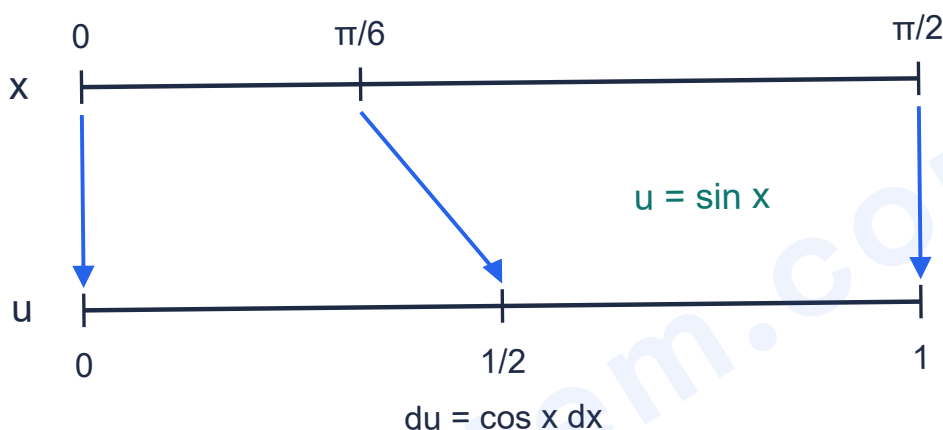


Fig. 1: with $u = \sin x$, the x -limits 0 , $\pi/6$ and $\pi/2$ map to the u -limits 0 , $1/2$ and 1 (drawn to scale on each line); a definite integral in u uses the new limits

Worked example: A linear substitution

Use the substitution $u = 2x + 1$ to find the exact value of $\int_0^4 x\sqrt{2x+1} dx$.

- $\frac{du}{dx} = 2$, so $dx = \frac{1}{2} du$; and $x = \frac{u-1}{2}$.
- Limits: $x = 0 \Rightarrow u = 1$; $x = 4 \Rightarrow u = 9$.
- The integral becomes $\int_1^9 \frac{u-1}{2} u^{1/2} \times \frac{1}{2} du = \frac{1}{4} \int_1^9 (u^{3/2} - u^{1/2}) du$.
- $= \frac{1}{4} \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_1^9 = \left[\frac{1}{10} u^{5/2} - \frac{1}{6} u^{3/2} \right]_1^9$.
- $9^{5/2} = 243$ and $9^{3/2} = 27$: $\frac{1}{10}(243 - 1) - \frac{1}{6}(27 - 1) = \frac{121}{5} - \frac{13}{3}$.

Answer: $\frac{298}{15}$

Integrand, dx, limits: convert all three.

- ✓ **Differentiate** the substitution and express dx (or a group like $\cos x \, dx$) in du .
- ✓ **Rewrite** any leftover x in terms of u .
- ✓ **Change the limits** for a definite integral.
- ✓ **Integrate** in u ; simplify powers first.
- ✓ **Back-substitute** for an indefinite integral.

Watch out!

Forgetting to replace dx is the most expensive slip: with $u = 2x + 1$, $dx = \frac{1}{2}du$, not du .

The second is mixing old and new limits: once the integral is in u , the limits must be u -values.

With $u = \cos x$, $du = -\sin x \, dx$. The minus sign is easy to drop; it is usually absorbed by swapping the limits, because $x = 0$ gives $u = 1$, the larger value.

An indefinite answer left in u loses the final mark. A definite answer evaluated with the old x -limits in a u -integral is simply wrong.

5 Integration by parts

Syllabus 3.5

$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$. Choose u to be the factor that gets simpler when differentiated (a power of x , or $\ln x$ and $\tan^{-1} x$, which you cannot integrate directly), and $\frac{dv}{dx}$ to be the factor you can integrate.

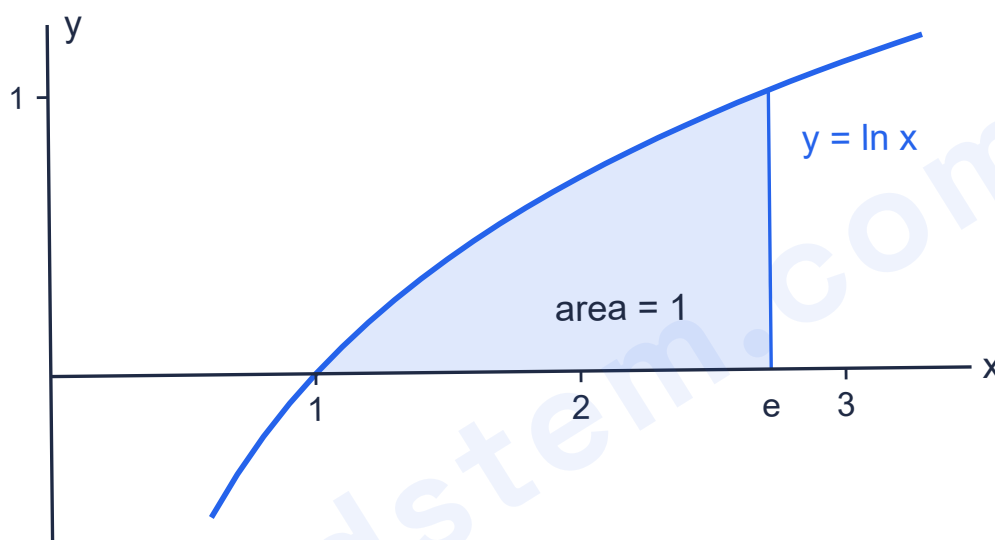


Fig. 1: the area under $y = \ln x$ from $x = 1$ to $x = e$ is found by parts with $u = \ln x$ and $dv/dx = 1$: $[x \ln x - x]$ from 1 to $e = (e - e) - (0 - 1) = 1$ square unit

Worked example: Power times exponential

Find $\int x e^{2x} dx$.

1. $u = x$, so $\frac{du}{dx} = 1$. $\frac{dv}{dx} = e^{2x}$, so $v = \frac{1}{2}e^{2x}$.

2. $uv - \int v \frac{du}{dx} dx = \frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x} dx$.

3. $\int \frac{1}{2}e^{2x} dx = \frac{1}{4}e^{2x}$.

Answer: $\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + c = \frac{1}{4}(2x - 1)e^{2x} + c$

Choose u so the new integral is simpler.

✓ **Formula:** $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$

✓ **u is** $\ln x$ or $\tan^{-1} x$ if present, otherwise the power of x .

✓ **Lone $\ln x$:** take $\frac{dv}{dx} = 1$.

✓ **Twice** for x^2 times an exponential or trig function; bracket the second result.

✓ **Check** by differentiating your answer.

Watch out!

Choosing $u = e^{2x}$ and $\frac{dv}{dx} = x$ makes things worse: the new integral contains $\frac{x^2}{2} e^{2x}$. If the integral you produce is harder than the original, swap u and $\frac{dv}{dx}$.

With $\ln x$ never choose $\frac{dv}{dx} = \ln x$: you would need to integrate $\ln x$ to find v , which is the very thing you cannot yet do.

The minus sign in front of the second integral applies to EVERYTHING it produces. When parts is used twice, bracket the second result: $-x^2 e^{-x} + (-2x e^{-x} - 2e^{-x})$. Sign slips here cost the accuracy marks.

6 The trapezium rule

Syllabus 3.5

With n intervals of width $h = \frac{b-a}{n}$ and ordinates $y_0, y_1, \dots, y_n$: $\int_a^b y \, dx \approx \frac{h}{2} [y_0 + 2(y_1 + \dots + y_{n-1}) + y_n]$. If the curve bends upwards (lies below its chords) the rule over-estimates; if it bends downwards it under-estimates.

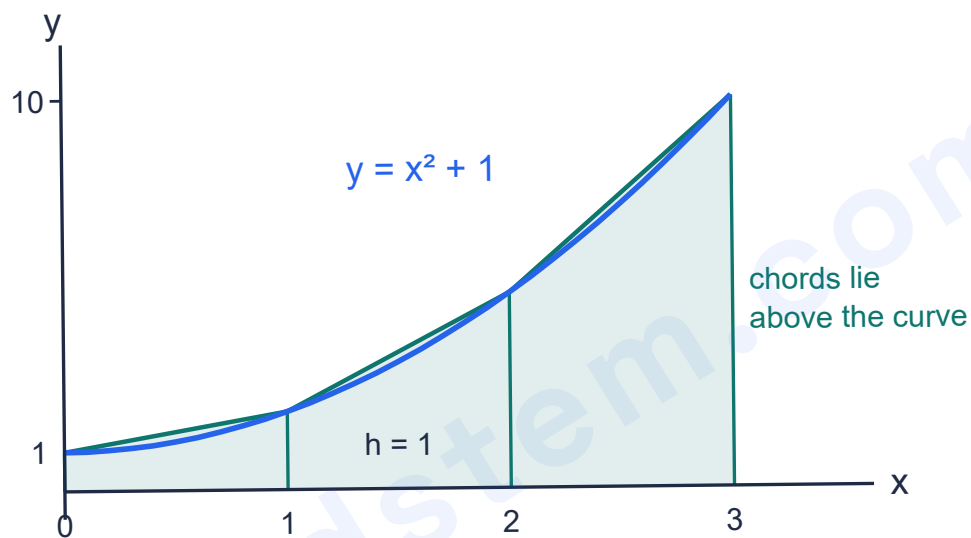


Fig. 1: three trapezia of width $h = 1$ under $y = x^2 + 1$ from $x = 0$ to $x = 3$; the curve bends upwards, so every chord lies above it and the estimate 12.5 is an over-estimate of the exact area 12

Worked example: A table of values

Use the trapezium rule with 4 intervals to estimate $\int_0^2 \sqrt{1+x^3} \, dx$, giving your answer to 3 significant figures.

- $h = \frac{2-0}{4} = 0.5$, so $x = 0, 0.5, 1, 1.5, 2$.
- $y = 1, 1.06066, 1.41421, 2.09165, 3$.
- Ends: $1 + 3 = 4$. Middles doubled: $2(1.06066 + 1.41421 + 2.09165) = 9.13304$.
- Estimate $= \frac{0.5}{2}(4 + 9.13304) = 0.25 \times 13.13304 = 3.28326$.

Answer: 3.28 (3 s.f.)

Width, table, weights, then a sketch.

- ✓ **Width:** $h = \frac{b - a}{n}$ with n intervals.
- ✓ **Table:** $n + 1$ ordinates to 4 or 5 significant figures; radians for trig.
- ✓ **Weights:** ends once, middles twice, times $\frac{h}{2}$.
- ✓ **Sketch:** bends up gives an over-estimate, bends down an under-estimate.
- ✓ **Round** only at the end, to the accuracy asked.

Watch out!

n intervals have $n + 1$ ordinates. With 4 intervals on $[0, 2]$, $h = 0.5$, not $\frac{2}{5}$. Equally, do not forget the $\frac{h}{2}$ at the end, or the answer comes out twice as big.

Do not judge by whether the curve is going up or down: $y = \ln x$ is increasing, but it bends downwards, so the rule under-estimates. It is the bending (concavity) that matters, not the slope.

In degree mode $\sin(0.785 \dots)$ is about 0.0137, not 0.707, and the estimate collapses. If a trigonometric estimate looks tiny, check the calculator mode.