

Trigonometry

A-Level Maths 9709 · Topic 3.3 revision notes

Everything from the 4 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Secant, cosecant and cotangent

2. Compound angle formulae

3. Double angle formulae

4. Expressing $a \sin x + b \cos x$ as $R \sin(x + a)$

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1 Secant, cosecant and cotangent

Syllabus 3.3

$\sec \theta = \frac{1}{\cos \theta}$, $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$, $\cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$. Dividing $\sin^2 \theta + \cos^2 \theta \equiv 1$ by $\cos^2 \theta$ or $\sin^2 \theta$ gives $1 + \tan^2 \theta \equiv \sec^2 \theta$ and $1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$.

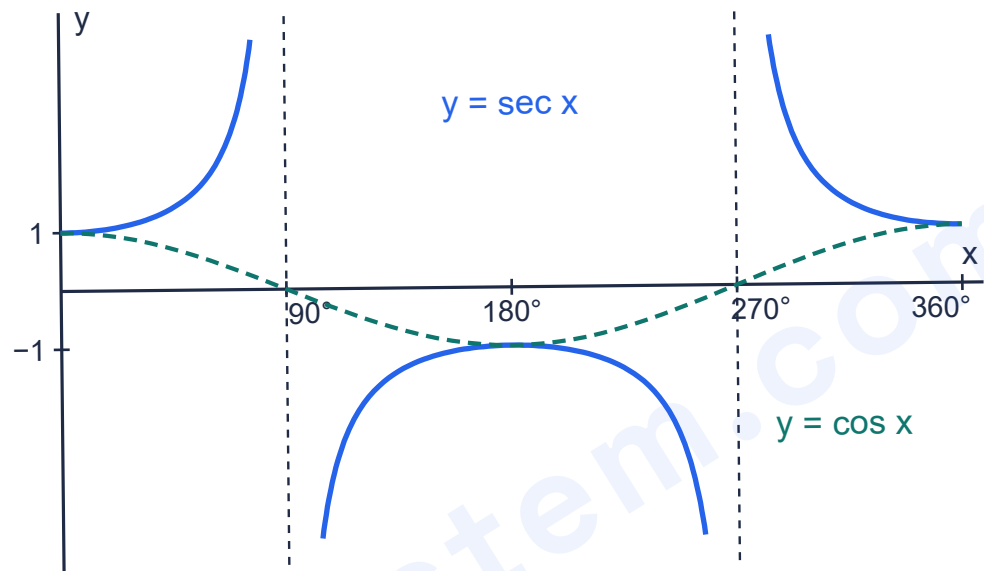


Fig. 1: $y = \sec x$ (blue) is $1 \div \cos x$ (dashed). Where $\cos x = \pm 1$ they touch; where $\cos x = 0$ $\sec x$ has vertical asymptotes, and $\sec x$ never lies between -1 and 1

Worked example: Exact values

Find the exact values of (a) $\sec 60^\circ$ (b) $\operatorname{cosec} 225^\circ$ (c) $\cot \frac{5\pi}{6}$.

- (a) $\cos 60^\circ = \frac{1}{2}$, so $\sec 60^\circ = 2$.
- (b) 225° is in the third quadrant, so $\sin 225^\circ = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$.
- $\operatorname{cosec} 225^\circ = -\sqrt{2}$.
- (c) $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\sin \frac{5\pi}{6} = \frac{1}{2}$, so $\cot \frac{5\pi}{6} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$.

Answer: (a) 2 (b) $-\sqrt{2}$ (c) $-\sqrt{3}$

Make every term the same function.

✓ **Definitions:** $\sec = \frac{1}{\cos}$, $\operatorname{cosec} = \frac{1}{\sin}$, $\cot = \frac{\cos}{\sin}$.

✓ **Identities:** $\sec^2 \theta \equiv 1 + \tan^2 \theta$,
 $\operatorname{cosec}^2 \theta \equiv 1 + \cot^2 \theta$.

✓ **Equations:** substitute so that only one function remains, then solve the quadratic.

✓ **Reject** impossible roots: $|\sec \theta| \geq 1$ and $|\operatorname{cosec} \theta| \geq 1$.

✓ **Proofs:** start from one side, convert to sin and cos, combine fractions, finish at the other side.

Watch out!

$\sec x$ is NOT $\cos^{-1} x$. The index -1 on a function name means the inverse function; $\sec x = (\cos x)^{-1}$ is the reciprocal. On the calculator, $\sec 50^\circ$ is $1 \div \cos 50^\circ = 1.56$, not $\cos^{-1} 50$ (which gives an error).

The identity is $\sec^2 \theta = 1 + \tan^2 \theta$, not $\sec^2 \theta = 1 - \tan^2 \theta$ or $\tan^2 \theta = 1 + \sec^2 \theta$. If you forget, re-derive it in one line by dividing $\sin^2 \theta + \cos^2 \theta = 1$ by $\cos^2 \theta$.

Never work on both sides at once, and never cross-multiply an identity you are asked to prove: that assumes the result. Write $LHS = \dots = RHS$ as one chain.

2 Compound angle formulae

Syllabus 3.3

$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$, $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$, $\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$. In the cosine formula the sign flips.

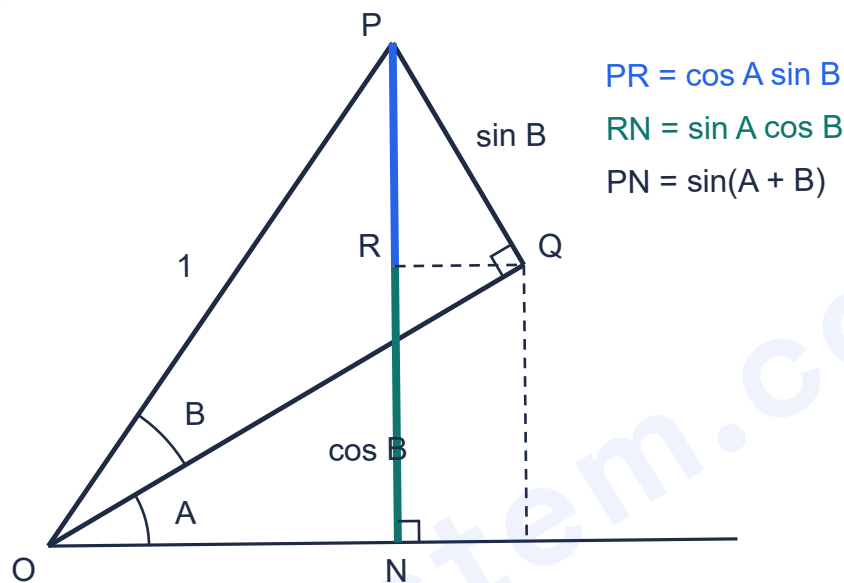


Fig. 1: $OP = 1$ makes angle $A + B$ with the horizontal (here $A = 30^\circ$, $B = 25^\circ$, drawn to scale). $OQ = \cos B$ and $QP = \sin B$. The height $PN = \sin(A + B)$ splits into $RN = \sin A \cos B$ and $PR = \cos A \sin B$, which is the formula for $\sin(A + B)$

Worked example: Exact value of $\sin 75^\circ$

Find the exact value of $\sin 75^\circ$.

1. Split into angles with known exact values: $75^\circ = 45^\circ + 30^\circ$.

2. $\sin 75^\circ = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$.

$$3. = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

Answer: $\sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$ (check: 0.966)

Expand with exact values, then collect.

- ✓ **Exact values:** split the angle into two from $30^\circ, 45^\circ, 60^\circ$ (or $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$).
- ✓ **Signs:** sine keeps the sign, cosine flips it; tangent keeps it on top and flips it underneath.
- ✓ **Given ratios:** find the missing ratios from a right-angled triangle, then fix signs by quadrant.
- ✓ **Equations:** expand, collect $\sin x$ and $\cos x$, divide to get $\tan x = k$.
- ✓ **Tan equations:** clear the fraction and solve the quadratic in $\tan x$.

Watch out!

$\sin(A + B) \neq \sin A + \sin B$.
Test it: $\sin(30^\circ + 60^\circ) = 1$,
but $\sin 30^\circ + \sin 60^\circ = 1.37$.
Brackets after $\sin$, $\cos$ or $\tan$
never multiply out.

Signs come from the quadrant, not from the triangle. If A is obtuse and $\sin A = \frac{5}{13}$, then $\cos A = -\frac{12}{13}$; forgetting the minus sign wrecks every later line.

Do not write $\sin(x + 30^\circ) = \sin x + \frac{1}{2}$. Expand properly: the known angle's sine AND cosine both appear.

The tangent formula has a 1 in the denominator: $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, with a MINUS sign for the plus case.

3 Double angle formulae

Syllabus 3.3

$\sin 2A \equiv 2 \sin A \cos A$, $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A$, $\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$. Choose the form of $\cos 2A$ that matches the other terms in the equation.

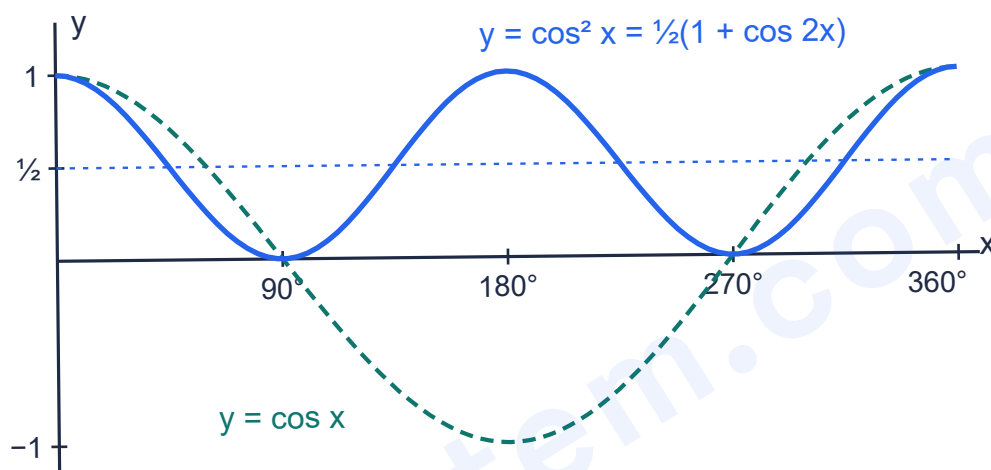


Fig. 1: $y = \cos^2 x$ (blue) never goes below 0. It oscillates about the line $y = \frac{1}{2}$ with amplitude $\frac{1}{2}$ and period 180° , exactly the graph of $\frac{1}{2} + \frac{1}{2} \cos 2x$, which is the identity $\cos 2x = 2\cos^2 x - 1$ rearranged

Worked example: All three from one ratio

Given that $\sin A = \frac{3}{5}$ and A is obtuse, find the exact values of $\sin 2A$, $\cos 2A$ and $\tan 2A$.

1. A is in the second quadrant, so $\cos A = -\frac{4}{5}$.
2. $\sin 2A = 2 \times \frac{3}{5} \times \left(-\frac{4}{5}\right) = -\frac{24}{25}$.
3. $\cos 2A = 1 - 2 \sin^2 A = 1 - 2 \times \frac{9}{25} = \frac{7}{25}$.
4. $\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{24}{7}$.

Answer: $\sin 2A = -\frac{24}{25}$, $\cos 2A = \frac{7}{25}$, $\tan 2A = -\frac{24}{7}$

Pick the form of $\cos 2x$ that fits.

- ✓ **sin:** $\sin 2A = 2 \sin A \cos A$.
- ✓ **cos:** $\cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$.
- ✓ **tan:** $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$.
- ✓ **Equations:** one angle, one function; factorise rather than divide.
- ✓ **Proofs:** $1 - \cos 2x = 2 \sin^2 x$ and $1 + \cos 2x = 2 \cos^2 x$ remove the 1.

Watch out!

$\sin 2A \neq 2 \sin A$. With $A = 90^\circ$: $\sin 180^\circ = 0$, but $2 \sin 90^\circ = 2$. The 2 belongs inside the function, so you need the formula.

Dividing both sides of $2 \sin x \cos x = \cos x$ by $\cos x$ throws away the solutions $\cos x = 0$ (here 90° and 270°). Bring everything to one side and factorise instead.

In a 'show that' question every step must be visible. Writing down $3 \sin x - 4 \sin^3 x$ straight after $\sin(2x + x)$ earns nothing.

4 Expressing $a \sin x + b \cos x$ as $R \sin(x + \alpha)$

Syllabus 3.3

$a \sin x + b \cos x \equiv R \sin(x + \alpha)$ where $R = \sqrt{a^2 + b^2}$, $R \cos \alpha = a$, $R \sin \alpha = b$, so $\tan \alpha = \frac{b}{a}$. Expand the target form, match coefficients of $\sin x$ and $\cos x$, then square-and-add for R and divide for α .

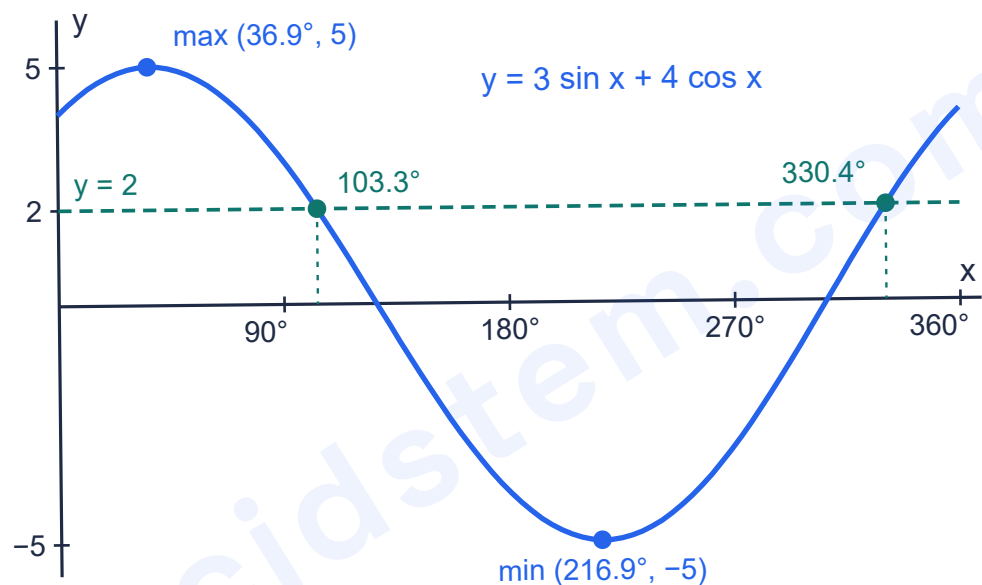


Fig. 1: $y = 3 \sin x + 4 \cos x$ is the single wave $y = 5 \sin(x + 53.1^\circ)$: amplitude 5, maximum at $x = 90^\circ - 53.1^\circ = 36.9^\circ$. It meets $y = 2$ at $x = 103.3^\circ$ and 330.4°

Worked example: $R \sin(x + \alpha)$

Express $3 \sin x + 4 \cos x$ in the form $R \sin(x + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$.

1. Expand: $R \sin(x + \alpha) = R \sin x \cos \alpha + R \cos x \sin \alpha$.
2. Match: $R \cos \alpha = 3$ and $R \sin \alpha = 4$.
3. $R = \sqrt{3^2 + 4^2} = 5$.
4. $\tan \alpha = \frac{4}{3}$, so $\alpha = 53.13^\circ$.

Answer: $3 \sin x + 4 \cos x \equiv 5 \sin(x + 53.13^\circ)$

Expand, match, square-and-add, divide.

- ✓ **Expand** the target form, e.g.
 $R \sin(x + \alpha) = R \sin x \cos \alpha + R \cos x \sin \alpha.$
- ✓ **Match** coefficients: $R \cos \alpha = \dots$,
 $R \sin \alpha = \dots$
- ✓ **R** = $\sqrt{a^2 + b^2}$; $\tan \alpha = \frac{R \sin \alpha}{R \cos \alpha}.$
- ✓ **Solve:** shift the range by α , find every value of the bracket, then undo the shift.
- ✓ **Max/min:** $-R \leq R \sin(x + \alpha) \leq R$; for a fraction, the greatest value uses the least denominator.

Watch out!

$\tan \alpha = \frac{R \sin \alpha}{R \cos \alpha}$, so it is the coefficient matched with $\sin \alpha$ on top. For $3 \sin x + 4 \cos x = R \sin(x + \alpha)$, $\tan \alpha = \frac{4}{3}$, not $\frac{3}{4}$. Always write the two matching equations first.

Solve for the whole bracket $x + \alpha$ in its shifted range FIRST, and only then subtract α . Taking the calculator value 23.58° and subtracting 53.13° gives a negative angle, and students often then miss 330.4° .

For a fraction with the trig expression underneath, the greatest value of the fraction comes from the LEAST value of the denominator. Also check the denominator can never be zero.