

Integration

A-Level Maths 9709 · Topic 1.8 revision notes

Everything from the 3 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Integration as reverse differentiation

2. Definite integrals and area

3. Volumes of revolution

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1 Integration as reverse differentiation

Syllabus 1.8

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$ and $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$, for any rational $n \neq -1$. A point on the curve fixes the value of c .

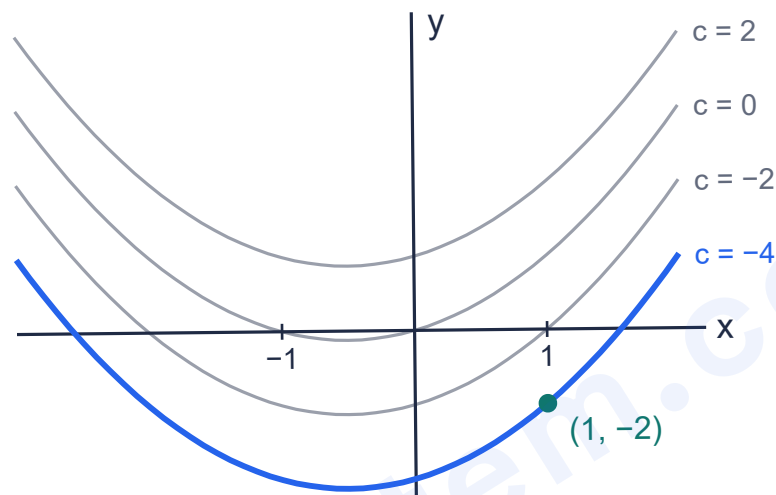


Fig. 1: every curve $y = x^2 + x + c$ has gradient $2x + 1$; only $c = -4$ passes through $(1, -2)$

Worked example: A polynomial

Find $\int (2x^3 - 5x + 1) dx$.

1. $2x^3 \rightarrow \frac{2x^4}{4} = \frac{x^4}{2}$.
2. $-5x \rightarrow -\frac{5x^2}{2}$, and $1 \rightarrow x$.
3. Add the constant of integration.

Answer: $\frac{x^4}{2} - \frac{5x^2}{2} + x + c$

Rewrite, integrate, add c, then fix c.

- ✓ **Rewrite:** roots and fractions as powers; split fractions term by term.
- ✓ **Powers:** $x^n \rightarrow \frac{x^{n+1}}{n+1}, n \neq -1.$
- ✓ **Linear bracket:** $(ax+b)^n \rightarrow \frac{(ax+b)^{n+1}}{a(n+1)}.$
- ✓ **Constant:** always +c; find it from a point on the curve.
- ✓ **Check:** differentiate your answer and compare.

Watch out!

Dividing by a fraction is multiplying by its reciprocal:

$$\frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}}, \text{ not } \frac{3}{2}x^{\frac{3}{2}}. \text{ And}$$

$\int_2^7 x^{-1} dx$ cannot be done this way (it would divide by zero).

Divide by a; do not multiply.

$$\int (2x+1)^3 dx \text{ is } \frac{(2x+1)^4}{8},$$

not $\frac{2(2x+1)^4}{4}$. This shortcut

only works when the inside is

LINEAR: $\int (x^2+1)^3 dx$ needs expanding.

Find c straight after integrating, using the equation of the CURVE (y), not the gradient. When you integrate twice, each integration needs its own constant.

2 Definite integrals and area

Syllabus 1.8

$\int_a^b f(x) dx = F(b) - F(a)$, where F is any integral of f . The area between $y = f(x)$ and the x -axis is $\int_a^b y dx$ when the curve is above the axis; between two graphs, integrate (top - bottom).

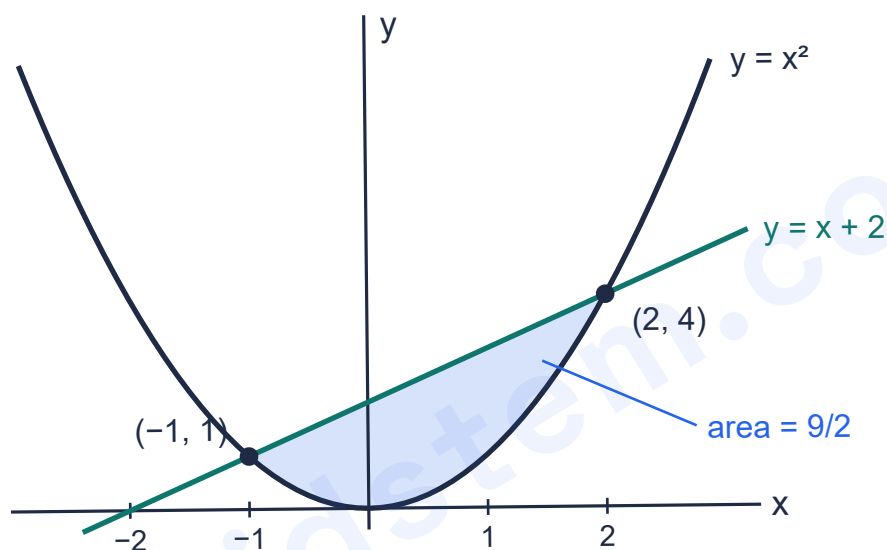


Fig. 1: the region between $y = x + 2$ and $y = x^2$, which meet at $(-1, 1)$ and $(2, 4)$, has area $9/2$

Worked example: A polynomial

Evaluate $\int_1^3 (3x^2 - 2x) dx$.

1. Integrate: $[x^3 - x^2]_1^3$.
2. Top limit: $27 - 9 = 18$. Bottom limit: $1 - 1 = 0$.
3. Subtract: $18 - 0 = 18$.

Answer: 18

Sketch, set limits, integrate, subtract.

- ✓ **Sketch:** where the region is, where the graphs meet, and whether it crosses the axis.
- ✓ **Limits:** from the question or from solving the equations simultaneously.
- ✓ **Integrate:** (upper — lower), or y for a region on the x -axis, or x in terms of y against the y -axis.
- ✓ **Evaluate:** $F(b) - F(a)$, with brackets around $F(a)$.
- ✓ **Below the axis:** integrate each part separately and add the sizes.

Watch out!

A lower limit of 0 does not always give 0: $(2 \times 0 + 1)^{\frac{3}{2}} = 1$.

Always substitute properly, and put the bottom-limit value in brackets before subtracting it.

Sketch first. If the curve crosses the x -axis between the limits, one integral gives (area above) — (area below), not the total area.

Upper minus lower. If you subtract the wrong way you get the negative of the area; an area is never negative, so take the size, but check your sketch.

An improper integral only has a value if the integrated expression settles to a limit. $[-\frac{1}{x}]$ at ∞ is 0, but something like $[2\sqrt{x}]$ at ∞ grows without limit, so

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx \text{ has no value.}$$

3 Volumes of revolution

Syllabus 1.8

About the x -axis: $V = \pi \int_a^b y^2 dx$. About the y -axis: $V = \pi \int_c^d x^2 dy$. For a region with a hole, subtract: $V = \pi \int_a^b (y_{\text{outer}}^2 - y_{\text{inner}}^2) dx$.

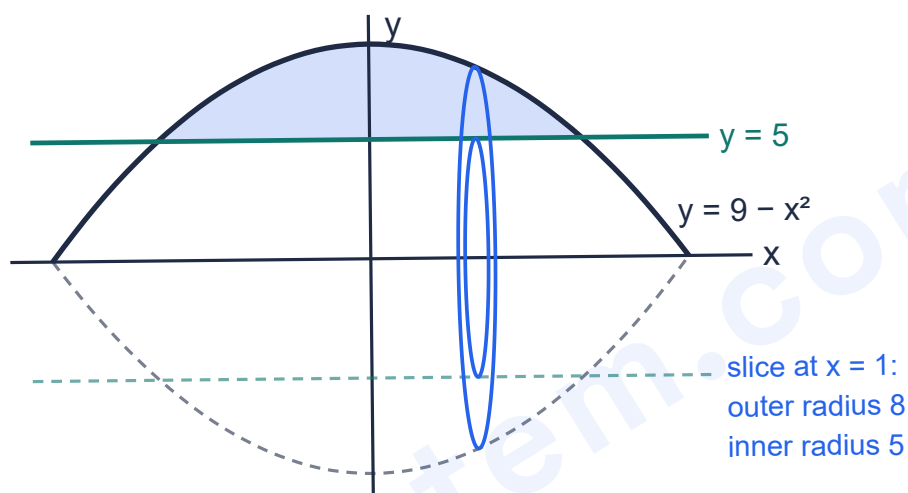


Fig. 1: the shaded region between $y = 9 - x^2$ and $y = 5$ rotated about the x -axis; each slice is a washer, outer radius $9 - x^2$ and inner radius 5 (8 and 5 at $x = 1$)

Worked example: A quadratic

The region between $y = x^2 + 1$, the x -axis, $x = 0$ and $x = 2$ is rotated through 360° about the x -axis. Find the exact volume.

1. Square first: $y^2 = (x^2 + 1)^2 = x^4 + 2x^2 + 1$.

2. $V = \pi \int_0^2 (x^4 + 2x^2 + 1) dx = \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x \right]_0^2$.

3. $= \pi \left(\frac{32}{5} + \frac{16}{3} + 2 \right) = \pi \times \frac{96+80+30}{15}$.

Answer: $\frac{206\pi}{15}$

Square the radius, then integrate along the axis.

- ✓ **About the x-axis:** $V = \pi \int_a^b y^2 dx$,
x-limits.
- ✓ **About the y-axis:** $V = \pi \int_c^d x^2 dy$,
y-limits.
- ✓ **Square first:** expand y^2 or write it as a power of a linear bracket.
- ✓ **Washers:** $\pi \int (\text{outer}^2 - \text{inner}^2)$.
- ✓ **Exact:** leave π in unless a decimal is asked for.

Watch out!

Square y BEFORE integrating:

$\pi \int y^2 dx$ is not $\pi \left(\int y dx \right)^2$

. And keep the π : an exact answer is usually wanted, such as $\frac{206\pi}{15}$.

About the y-axis, the limits are y -values. Using $x = 1$ to $x = 2$

(where the curve meets $y = 1$ and $y = 4$) with $\int x^2 dy$ mixes the two variables.

Washers: subtract the SQUARES,

$\pi \int (y_1^2 - y_2^2) dx$. Squaring the difference, $\pi \int (y_1 - y_2)^2 dx$, rotates the wrong shape and is the most common error.