

Logarithmic and exponential functions

A-Level Maths 9709 · Topic 3.2 revision notes

Everything from the 4 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Logarithms and the laws of logarithms
2. The exponential function and natural logarithms
3. Solving exponential equations and inequalities
4. Reducing relationships to linear form

1 Logarithms and the laws of logarithms

Syllabus 3.2

$\log_a x = y \iff a^y = x$ (for $a > 0$, $a \neq 1$, $x > 0$). The laws: $\log_a xy = \log_a x + \log_a y$, $\log_a \frac{x}{y} = \log_a x - \log_a y$, $\log_a x^k = k \log_a x$. Always check that every logarithm in the original equation is of a positive number.

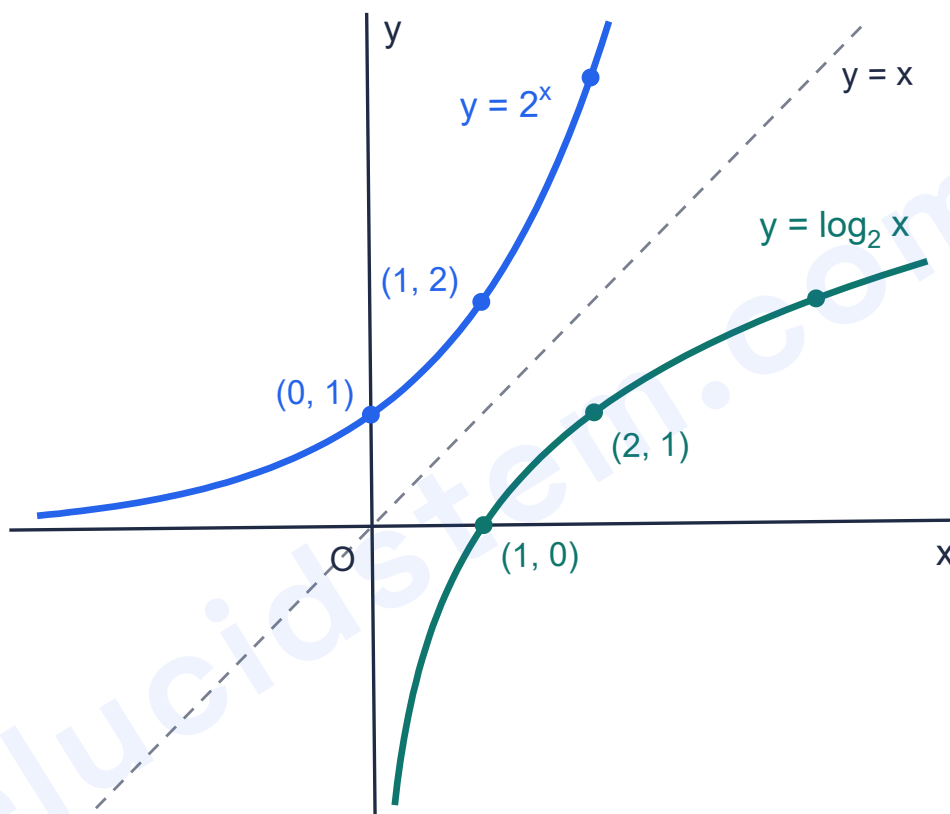


Fig. 1: $y = \log_2 x$ is the reflection of $y = 2^x$ in the line $y = x$. Each point (p, q) on one graph matches (q, p) on the other, e.g. $(1, 2)$ and $(2, 1)$

Worked example: Switching forms

(a) Write $\log_3 81 = 4$ in index form. (b) Write $2^{-3} = \frac{1}{8}$ in log form.

1. (a) The base 3 raised to the answer 4 gives 81.

2. (b) The base is 2, the power is -3 , the result is $\frac{1}{8}$.

Answer: (a) $3^4 = 81$ (b) $\log_2 \frac{1}{8} = -3$

One log, then index form, then check.

✓ **Definition:** $\log_a x = y \iff a^y = x$

✓ **Power law first:** move coefficients inside: $k \log_a x = \log_a x^k$.

✓ **Combine:** sums become products, differences become quotients.

✓ **Undo the log:** $\log_a(\text{stuff}) = c$ means $\text{stuff} = a^c$.

✓ **Check:** reject any root that makes the inside of an original log zero or negative.

Watch out!

$\log_2 \frac{1}{8}$ is -3 , not $\frac{1}{3}$ or $-\frac{1}{3}$.

Negative logs come from fractions; fractional logs come from roots.

There is no law for $\log(x + y)$:

$\log_a(x + y) \neq \log_a x + \log_a y$

. Also $\frac{\log_a x}{\log_a y}$ is NOT $\log_a \frac{x}{y}$.

Do not 'cancel the logs' term by term: $\log_2(x + 3) + \log_2(x -$

$1) = 5$ does NOT give $(x + 3) + (x - 1) = 5$ or $= 32$.

Make ONE log first, then undo it.

2 The exponential function and natural logarithms

Syllabus 3.2

$\ln x = \log_e x$, so $y = e^x \iff x = \ln y$. They undo each other: $\ln(e^x) = x$ for all x , and $e^{\ln x} = x$ for $x > 0$. $y = e^{kx}$ always passes through $(0, 1)$, stays above the x -axis, grows for $k > 0$ and decays towards $y = 0$ for $k < 0$.

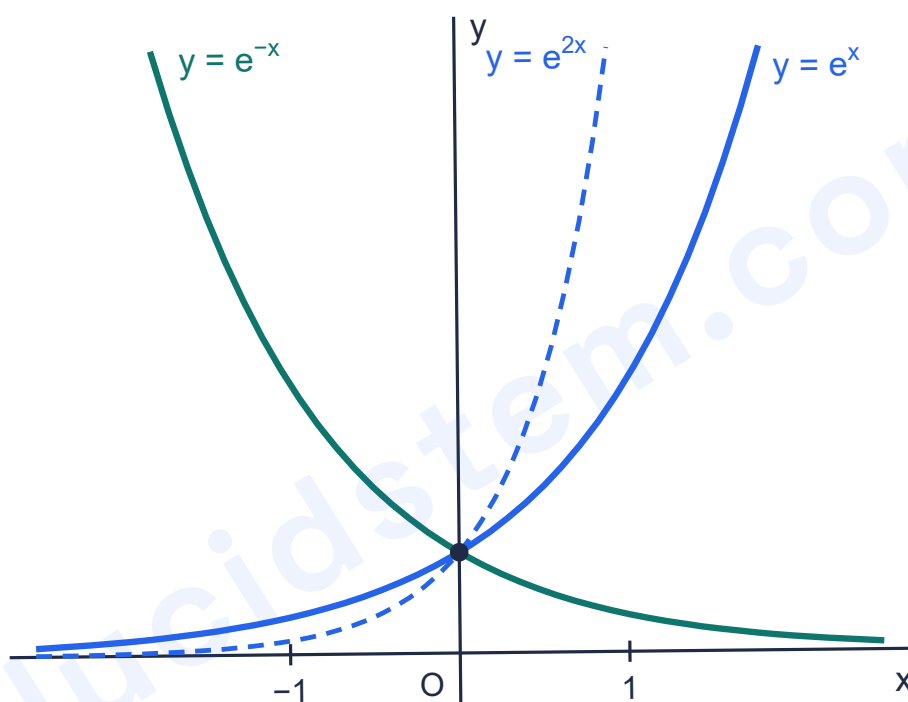


Fig. 1: $y = e^x$, $y = e^{2x}$ (dashed, steeper) and $y = e^{-x}$ (the reflection of $y = e^x$ in the y -axis). All three pass through $(0, 1)$ and approach the x -axis without reaching it

Worked example: Sketching a transformed exponential

Sketch $y = 4 - 3e^{-x}$, giving the exact coordinates of any points where it meets the axes and the equation of the asymptote.

1. $x = 0$: $y = 4 - 3 = 1$, so $(0, 1)$.
2. As $x \rightarrow \infty$, $e^{-x} \rightarrow 0$, so $y \rightarrow 4$: the asymptote is $y = 4$, approached from below.
3. $y = 0$: $3e^{-x} = 4$, so $e^{-x} = \frac{4}{3}$ and $-x = \ln \frac{4}{3}$.
4. So $x = -\ln \frac{4}{3} = \ln \frac{3}{4} \approx -0.288$. As $x \rightarrow -\infty$, $y \rightarrow -\infty$.

Answer: Increasing curve through $\left(\ln \frac{3}{4}, 0\right)$ and $(0, 1)$, levelling off below the asymptote $y = 4$.

Isolate, then apply the inverse.

- ✓ **Inverse pair:** $\ln(e^x) = x$ and $e^{\ln x} = x$ ($x > 0$).
- ✓ **Isolate first:** get $e^{(\dots)}$ or $\ln(\dots)$ alone on one side.
- ✓ **Quadratic in e^x :** let $u = e^x$; reject $u \leq 0$.
- ✓ **Sketch:** y -intercept, asymptote $y = c$ for $Ae^{kx} + c$, and any x -intercept.
- ✓ **Exact or 3 s.f.:** give $\ln$ forms when asked for exact answers.

Watch out!

e^{-x} is never negative: $y = e^{-x}$ is a decreasing curve ABOVE the x -axis, not the curve $y = e^x$ flipped below it. The minus sign in the power reflects in the y -axis; a minus sign in front, $-e^x$, reflects in the x -axis.

$\ln(e^x + 1)$ is NOT $x + 1$, and $\ln(2e^x)$ is $\ln 2 + x$, not $2x$. $\ln$ only undoes e when the whole inside of the log is a single power of e .

Take $\ln$ only after isolating the exponential: from $3e^x + 1 = 10$, first get $e^x = 3$, then $x = \ln 3$. Writing $\ln 3 + x + \ln 1 = \ln 10$ is not a valid step, because there is no law for the log of a sum.

3 Solving exponential equations and inequalities

Syllabus 3.2

Take $\ln$ (or $\lg$) of both sides: $a^{f(x)} = b \Rightarrow f(x) \ln a = \ln b$. For an inequality, dividing by $\ln a$ keeps the sign when $a > 1$ but REVERSES it when $0 < a < 1$, because then $\ln a < 0$.

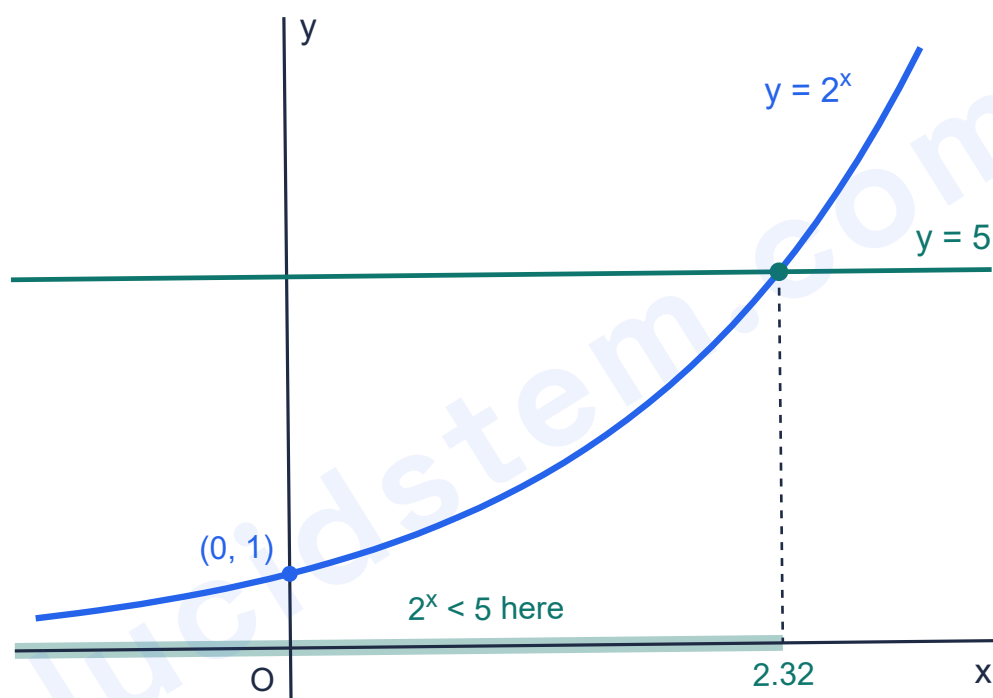


Fig. 1: $y = 2^x$ crosses $y = 5$ where $x = \ln 5 / \ln 2 = 2.32$, and because 2^x is increasing, $2^x < 5$ exactly when $x < 2.32$

Worked example: Simple index equation

Solve $2^x = 5$, giving your answer to 3 significant figures.

1. Take $\ln$: $x \ln 2 = \ln 5$.
2. $x = \frac{\ln 5}{\ln 2} = 2.3219 \dots$

Answer: $x = 2.32$ (Fig. 1)

Isolate, take logs, solve, check the sign.

- ✓ **Isolate** the power $a^{f(x)}$ on one side.
- ✓ **Take $\ln$** of both sides: $f(x) \ln a = \ln b$.
- ✓ **Both sides:** expand, collect x terms, factorise out x .
- ✓ **Inequalities:** reverse the sign when dividing by $\ln a$ with $0 < a < 1$.
- ✓ **Hidden quadratic:** $u = a^x$, reject $u \leq 0$.
- ✓ **Integers:** for 'smallest n ', round in the direction the inequality demands.

Watch out!

$\frac{\ln 5}{\ln 2}$ is NOT $\ln \frac{5}{2}$ and not $\ln 5 - \ln 2$. A quotient of logs does not simplify: work it out as one number divided by another.

In $n \ln 0.8 < \ln 0.1$, both logs are negative. Forgetting to reverse gives $n < 10.3$, which says the SMALL values work, the opposite of what happens as 0.8^n shrinks.

2^{2x+1} is $2 \times 2^{2x} = 2u^2$, not $u^2 + 1$ and not $2u$. Split the index with $a^{m+n} = a^m \times a^n$ before you substitute.

4 Reducing relationships to linear form

Syllabus 3.2

$y = kx^n \Rightarrow \ln y = \ln k + n \ln x$: plot $\ln y$ against $\ln x$; gradient n , intercept $\ln k$.

$y = ka^x \Rightarrow \ln y = \ln k + x \ln a$: plot $\ln y$ against x ; gradient $\ln a$, intercept $\ln k$.

Compare with $Y = mX + c$.

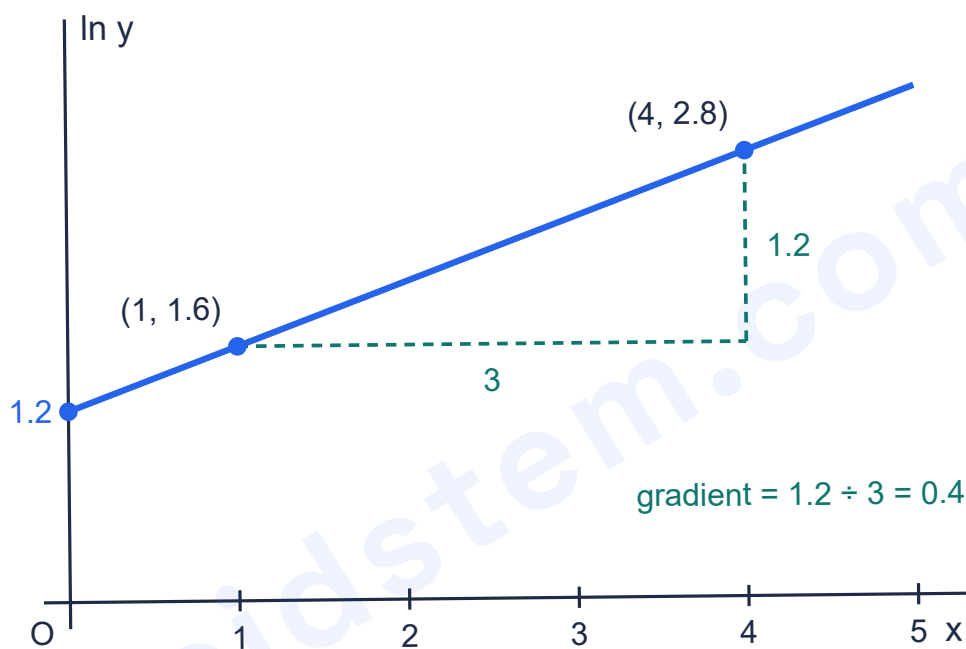


Fig. 1: for $y = A b^x$, the graph of $\ln y$ against x is a straight line. Here it passes through $(1, 1.6)$ and $(4, 2.8)$, so the gradient $\ln b = 0.4$ and the intercept $\ln A = 1.2$

Worked example: Choosing the axes

The variables satisfy $y = 5x^3$. Show that a suitable graph is a straight line, and state its gradient and intercept.

1. Take $\ln$: $\ln y = \ln(5x^3) = \ln 5 + \ln x^3$.

2. Power law: $\ln y = 3 \ln x + \ln 5$.

3. This is $Y = mX + c$ with $Y = \ln y$, $X = \ln x$, $m = 3$, $c = \ln 5$.

Answer: $\ln y$ against $\ln x$: a straight line with gradient 3 and intercept $\ln 5 \approx 1.61$

Take logs, match $Y = mX + c$, exponentiate.

- ✓ $y = kx^n$: $\ln y$ against $\ln x$; gradient n , intercept $\ln k$.
- ✓ $y = ka^x$: $\ln y$ against x ; gradient $\ln a$, intercept $\ln k$.
- ✓ **Gradient** from two points, then **intercept** from $Y = mX + c$ at one point.
- ✓ **Undo the logs**: $k = e^c$, $a = e^m$ (or $10^{(\dots)}$ for $\lg$).
- ✓ **Answer** the question: write the model, then substitute if asked.

Watch out!

For $y = ka^x$ the horizontal axis is plain x , not $\ln x$: the x is already in the index, and taking logs brings it down as it is. Plot $\ln x$ only when x is raised to a constant power.

The coordinates in these questions are values of $\ln y$ (and perhaps $\ln x$), not of y . The intercept of the line is $\ln k$; answering $k = 2.6$ instead of $k = e^{2.6}$ loses the final marks.

Undo with the SAME base you took logs in. If the axis is $\lg y$, use $10^{1.2}$; using $e^{1.2}$ gives 3.32, a completely different (wrong) constant.