

Numerical solution of equations

A-Level Maths 9709 · Topic 3.6 revision notes

Everything from the 2 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Locating roots by a change of sign

2. Iterative methods

1 Locating roots by a change of sign

Syllabus 3.6

Write the equation as $f(x) = 0$. If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then there is at least one root between a and b . To show a root is α correct to 2 d.p., show the sign changes between $\alpha - 0.005$ and $\alpha + 0.005$.

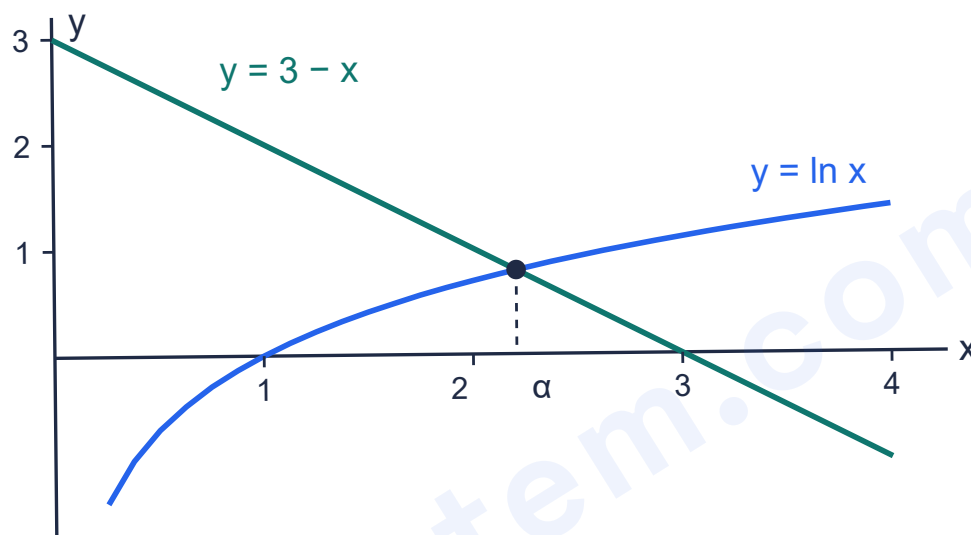


Fig. 1: $y = \ln x$ and $y = 3 - x$ cross exactly once, so $\ln x = 3 - x$ has one root α , between 2 and 3 ($\alpha \approx 2.21$)

Worked example: Graphs and a sign change

By sketching suitable graphs, show that the equation $\ln x = 3 - x$ has exactly one real root. Verify by calculation that this root lies between 2 and 3.

1. Sketch $y = \ln x$ (increasing, through $(1, 0)$) and $y = 3 - x$ (decreasing straight line). An increasing curve and a decreasing line cross exactly once (Fig. 1).
2. Let $f(x) = \ln x + x - 3$, continuous for $x > 0$.
3. $f(2) = \ln 2 - 1 = -0.307$ and $f(3) = \ln 3 = 1.099$.
4. The sign changes and f is continuous, so the root lies between 2 and 3.

Answer: One root, $2 < \alpha < 3$

Rearrange to $f(x) = 0$, then test the sign.

- ✓ **Sketch** both sides of the equation to count roots and see roughly where they are.
- ✓ **Rearrange** to $f(x) = 0$ and check f is continuous on the interval.
- ✓ **Evaluate** $f(a)$ and $f(b)$, in radians for trig, and quote the values.
- ✓ **Conclude:** "sign change, and f continuous, so a root lies between a and b ".
- ✓ **Accuracy:** test the rounding bounds, e.g. 1.505 and 1.515 for 1.51.

Watch out!

In a sketch-and-count question the marks are for two correct graph shapes in the right places AND a stated conclusion ("one point of intersection, so one root"). A sketch with no statement does not earn the final mark.

Compare with zero, not with each other: you must rearrange to $f(x) = 0$ first. Checking that $\ln 2 < 3 - 2$ and $\ln 3 > 3 - 3$ is fine, but only if you say clearly that the sign of $\ln x - (3 - x)$ has changed.

No sign change does NOT mean no root. $f(x) = (x - 1)^2$ has a root at $x = 1$ but is positive on both sides; and two roots between a and b cancel out the sign change. A sketch protects you from both traps.

2 Iterative methods

Syllabus 3.6

If the sequence given by $x_{n+1} = F(x_n)$ converges to α , then $\alpha = F(\alpha)$: the limit is a root of the equation $x = F(x)$. Iterate until successive values agree to the accuracy required, then give the root to that accuracy.

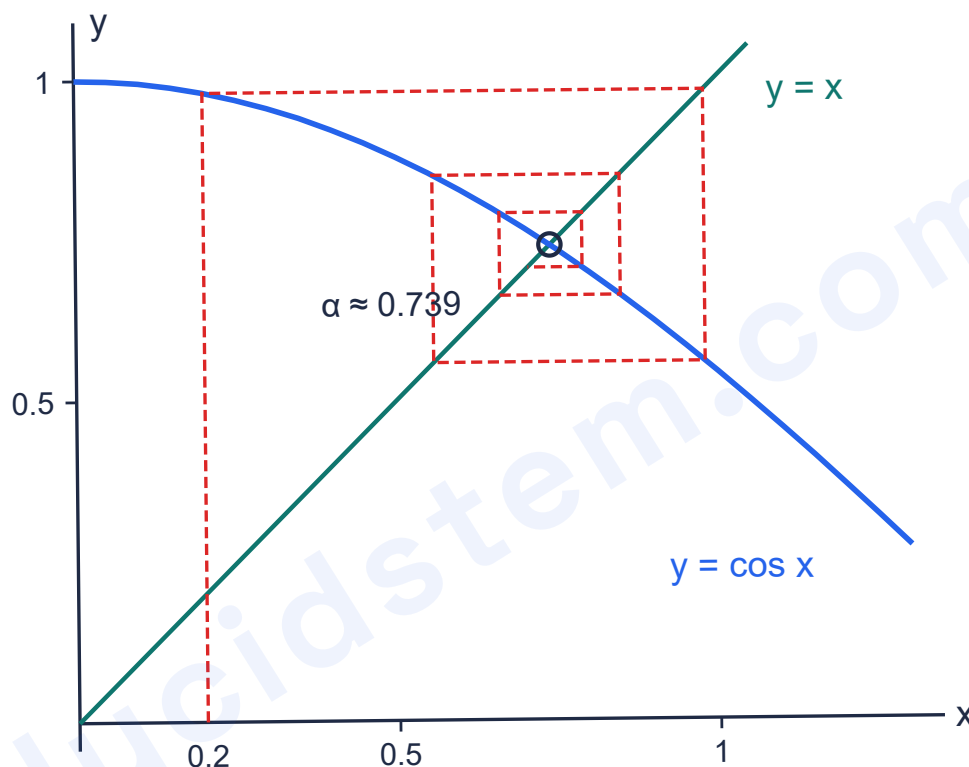


Fig. 1: a cobweb diagram for $x_{n+1} = \cos x_n$ starting at $x_1 = 0.2$: go up to the curve, across to $y = x$, up or down to the curve again; the path spirals in to the crossing point $\alpha \approx 0.739$, the root of $x = \cos x$

Worked example: Find the exact limit

The sequence defined by $x_{n+1} = \frac{2x_n}{3} + \frac{4}{3x_n^2}$, $x_1 = 1.5$, converges to α . Find the exact value of α .

1. At the limit, $\alpha = \frac{2\alpha}{3} + \frac{4}{3\alpha^2}$.
2. Subtract $\frac{2\alpha}{3}$: $\frac{\alpha}{3} = \frac{4}{3\alpha^2}$.
3. Multiply by $3\alpha^2$: $\alpha^3 = 4$.

Answer: $\alpha = \sqrt[3]{4}$

Rearrange, iterate, stop, check.

- ✓ **Rearrange** the equation into $x = F(x)$, or check that a given F comes from it.
- ✓ **Iterate** with ANS, writing each value to one more decimal place than needed.
- ✓ **Stop** when successive values agree to the required accuracy.
- ✓ **Exact limit:** put $x_{n+1} = x_n = \alpha$ and solve.
- ✓ **Failure:** values that run away or oscillate more widely mean the rearrangement does not converge.

Watch out!

Keep x_{n+1} and x_n distinct while iterating, but set them BOTH equal to α (or x) when finding the equation. Writing $\alpha = \frac{2x}{3} + \dots$ with a mixture of letters goes nowhere.

Do not round each iterate before feeding it back in: use the full calculator value (ANS). Rounding early can stall the sequence or change the final digit.

A start close to the root does not guarantee convergence: the rearrangement decides. In an exam, if your values run away, recheck the arithmetic and calculator mode (radians!) before concluding that the iteration fails.