

Permutations and combinations

A-Level Maths 9709 · Topic 5.2 revision notes

Everything from the 2 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

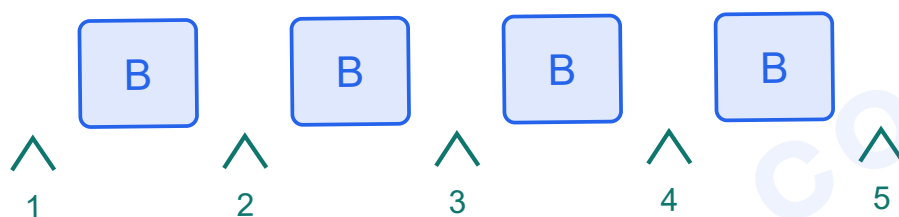
1. Permutations: arrangements in a line and with restrictions
2. Combinations: selections

1 Permutations: arrangements in a line and with restrictions

Syllabus 5.2

n different objects can be arranged in a line in $n!$ ways. Divide by $p!$ for each group of p identical objects. Glue objects that must be together into one unit; for objects that must be apart, subtract the 'together' count from the total or slot them into gaps.

Arrange the 4 boys first: $4!$ ways



Then put the 3 girls into 3 of the 5 gaps: $5 \times 4 \times 3$ ways

Fig. 1: the gaps method. No two girls can be next to each other if each girl takes a different gap between or beside the boys: $4! \times 5 \times 4 \times 3 = 1440$ arrangements

Worked example: Repeated letters

Find the number of different arrangements of the 8 letters of the word NEEDLESS.

1. The letters are N, E, E, E, D, L, S, S: three Es and two Ss.
2. If all were different there would be $8! = 40\,320$ arrangements.
3. Swapping the Es among themselves ($3!$ ways) or the Ss ($2!$ ways) gives the same word, so divide.

$$4. \frac{8!}{3! 2!} = \frac{40\,320}{12}.$$

Answer: 3360

Restrictions first, then fill the rest.

- ✓ **All different:** $n!$; only r places: ${}^nP_r$.
- ✓ **Repeats:** divide by $p!$ for every group of p identical objects.
- ✓ **Together:** glue into one unit, multiply by the arrangements inside.
- ✓ **Two apart:** total minus together.
Several apart: gaps method.
- ✓ **Overlapping restrictions:** split into cases and add.

Watch out!

Dividing by $3! + 2!$ instead of $3! \times 2!$. Each repeated group contributes its own factorial to the product in the denominator.

Forgetting the internal arrangements of the block. With different objects inside, multiply by their factorial; with identical objects inside (EEE), there is nothing to multiply.

Using 'total minus together' for three or more people who must all be apart. Subtracting 'all three together' leaves arrangements where two of them are still next to each other. Use the gaps method instead.

When two restrictions overlap (first digit and last digit both even), split into cases. A single product $3 \times 3 \times 4 \times 3$ double-uses digits like 4 and 6.

2 Combinations: selections

Syllabus 5.2

The number of ways of choosing r objects from n different objects, when order does not matter, is ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$. Since each selection can be ordered in $r!$ ways, ${}^nP_r = {}^nC_r \times r!$.

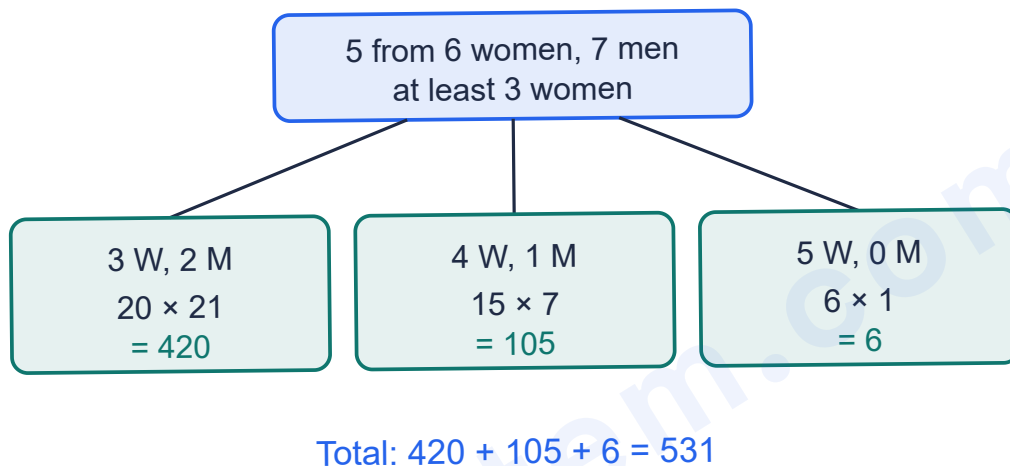


Fig. 1: 'at least 3 women' splits into three separate cases. Within a case multiply (women AND men); across cases add (one case OR another)

Worked example: A simple selection

In how many ways can a team of 4 be chosen from 10 players?

1. Order does not matter, so use ${}^{10}C_4$.

$$2. \frac{10!}{4!6!} = \frac{10 \times 9 \times 8 \times 7}{24}$$

Answer: 210

Read the condition, then choose a strategy.

- ✓ **Order irrelevant:** ${}^nC_r$; order relevant: ${}^nP_r = {}^nC_r \times r!$.
- ✓ **From two groups:** multiply the choices from each group.
- ✓ **At least / at most:** list the cases and add, or total minus forbidden cases.
- ✓ **Particular people:** include them and choose the rest, or remove them from the pool.
- ✓ **Equal unnamed groups:** divide by the number of ways to swap them.

Watch out!

Using ${}^nP_r$ for a committee.
Ordered counting counts each committee $r!$ times over: ${}^{10}P_4 = 5040$ is 24 times too big.

Counting 'at least one woman' as ${}^4C_1 \times {}^8C_3$: choose one woman, then any 3 of the rest. A group with two women is then counted twice, once for each woman picked 'first'.

Subtracting only the committees with neither Khan (462). The forbidden case is BOTH Khans, not neither.

Fixing C in the first position (${}^7P_3 = 210$). 'Includes C' means C may be in any of the four places.