

Quadratics

A-Level Maths 9709 · Topic 1.1 revision notes

Everything from the 5 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Completing the square

2. The discriminant

3. Quadratic equations and inequalities

4. Simultaneous equations: one linear, one quadratic

5. Equations quadratic in a function of x

thelucidstem.com

1 Completing the square

Syllabus 1.1

$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$. In the form $a(x + p)^2 + q$, the vertex is $(-p, q)$: a minimum if $a > 0$, a maximum if $a < 0$.

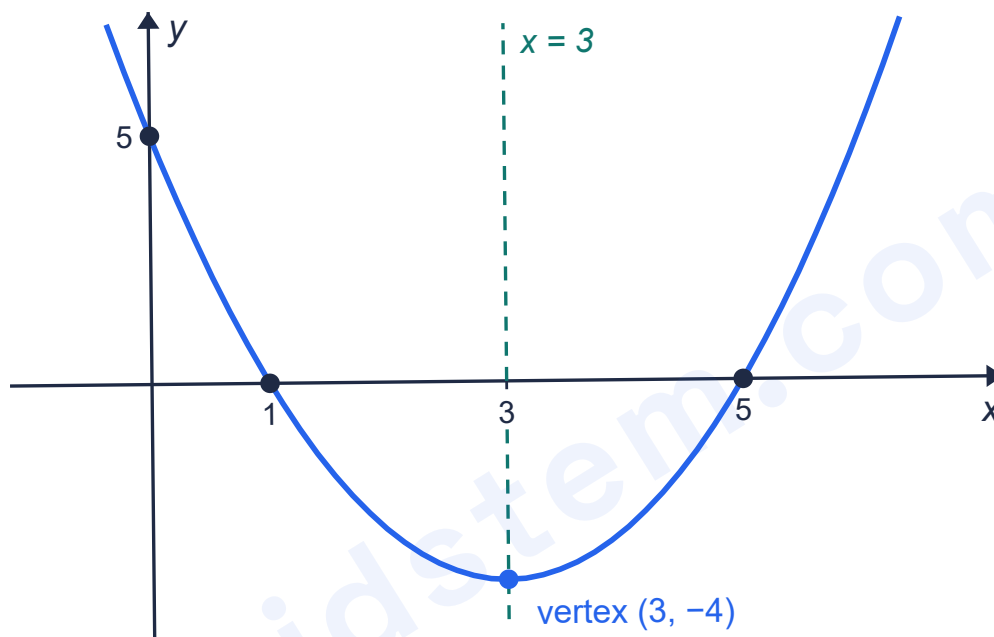


Fig. 1: $y = x^2 - 6x + 5 = (x - 3)^2 - 4$ has its vertex at $(3, -4)$, line of symmetry $x = 3$, roots 1 and 5 and y-intercept 5

Worked example: Integer half

Express $x^2 - 6x + 11$ in the form $(x - a)^2 + b$.

1. Half of -6 is -3 , so start with $(x - 3)^2 = x^2 - 6x + 9$.
2. That bracket contains an extra 9 we did not have, so subtract it: $x^2 - 6x = (x - 3)^2 - 9$.
3. Add the constant: $(x - 3)^2 - 9 + 11 = (x - 3)^2 + 2$.

Answer: $(x - 3)^2 + 2$, so $a = 3$, $b = 2$

Halve, square, subtract, tidy.

- ✓ **Factor** a out of the x^2 and x terms if $a \neq 1$.
- ✓ **Halve** the coefficient of x inside the bracket: $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}$.
- ✓ **Multiply back** by a , including the subtracted square, then collect constants.
- ✓ **Read off** $a(x + p)^2 + q$: vertex $(-p, q)$, minimum if $a > 0$, maximum if $a < 0$.
- ✓ **Check** by expanding: you must get the original quadratic back.

Watch out!

The sign inside the bracket matches the sign of b : $x^2 - 6x$ gives $(x - 3)^2$, not $(x + 3)^2$. And the square you subtract is always positive: $-(-3)^2 = -9$.

Writing $2x^2 - 12x + 7 = 2(x - 3)^2 - 9 + 7$ is the classic slip: the 9 sits *INSIDE* the bracket that is multiplied by 2, so it must become 18. Expand your answer to catch it.

The vertex of $a(x + p)^2 + q$ is at $x = -p$, not $x = p$: $(x + 1)^2$ is zero when $x = -1$. Read the x -coordinate as the value that makes the bracket zero.

2 The discriminant

Syllabus 1.1

For $ax^2 + bx + c = 0$: if $b^2 - 4ac > 0$ there are two distinct real roots; if $b^2 - 4ac = 0$ there is one repeated root; if $b^2 - 4ac < 0$ there are no real roots. 'Real roots' (not necessarily distinct) means $b^2 - 4ac \geq 0$.

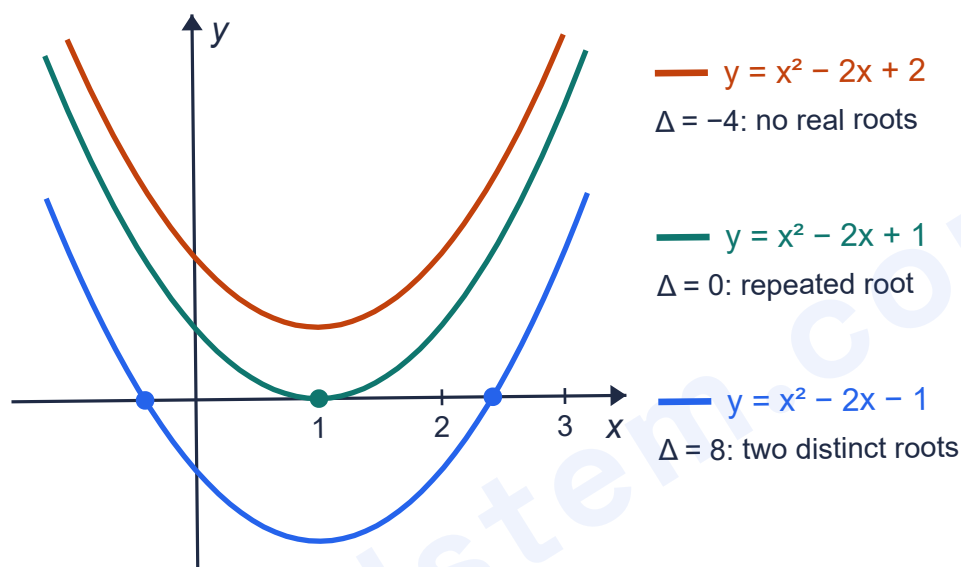


Fig. 1: three parabolas with the same shape, where $\Delta = b^2 - 4ac$. The curve with $\Delta > 0$ crosses the x-axis twice, $\Delta = 0$ touches it once, $\Delta < 0$ misses it

Worked example: Classify three equations

How many real roots does each equation have? (a) $3x^2 - 5x + 1 = 0$ (b) $4x^2 - 12x + 9 = 0$ (c) $2x^2 + 3x + 4 = 0$

- (a) $a = 3$, $b = -5$, $c = 1$: $b^2 - 4ac = 25 - 12 = 13 > 0$, so two distinct real roots.
- (b) $a = 4$, $b = -12$, $c = 9$: $144 - 144 = 0$, so one repeated root.
- Here $4x^2 - 12x + 9 = (2x - 3)^2$, so the repeated root is $x = \frac{3}{2}$.
- (c) $a = 2$, $b = 3$, $c = 4$: $9 - 32 = -23 < 0$, so no real roots.

Answer: (a) two (b) one repeated root, $x = \frac{3}{2}$ (c) none

Rearrange, read a , b , c , then decide.

- ✓ **Rearrange** to $ax^2 + bx + c = 0$ and read a , b , c with their signs.
- ✓ **Translate** the words: equal $= 0$; distinct > 0 ; real ≥ 0 ; no real < 0 .
- ✓ **Substitute** with brackets: $(b)^2 - 4(a)(c)$.
- ✓ **Solve** the resulting equation or inequality in k ; a quadratic inequality has a two-ended answer.
- ✓ **Always positive** needs $a > 0$ as well as $b^2 - 4ac < 0$.

Watch out!

b^2 is never negative: with $b = -5$, $b^2 = 25$, not -25 . Bracket negative values when you substitute: $(-5)^2 - 4(3)(1)$.

Read the wording exactly. 'Real roots' includes equal roots, so it needs ≥ 0 ; 'two distinct real roots' needs strict > 0 . Using the wrong one costs the final mark.

For 'positive for all x ' the condition $b^2 - 4ac < 0$ is not enough on its own: $-x^2 - 1$ has a negative discriminant but is always NEGATIVE. You also need $a > 0$.

3 Quadratic equations and inequalities

Syllabus 1.1

To solve a quadratic inequality, first solve the equation to find the critical values, then sketch the parabola. For $a > 0$: $(x - \alpha)(x - \beta) < 0$ gives $\alpha < x < \beta$ (between the roots); $(x - \alpha)(x - \beta) > 0$ gives $x < \alpha$ or $x > \beta$ (outside them).

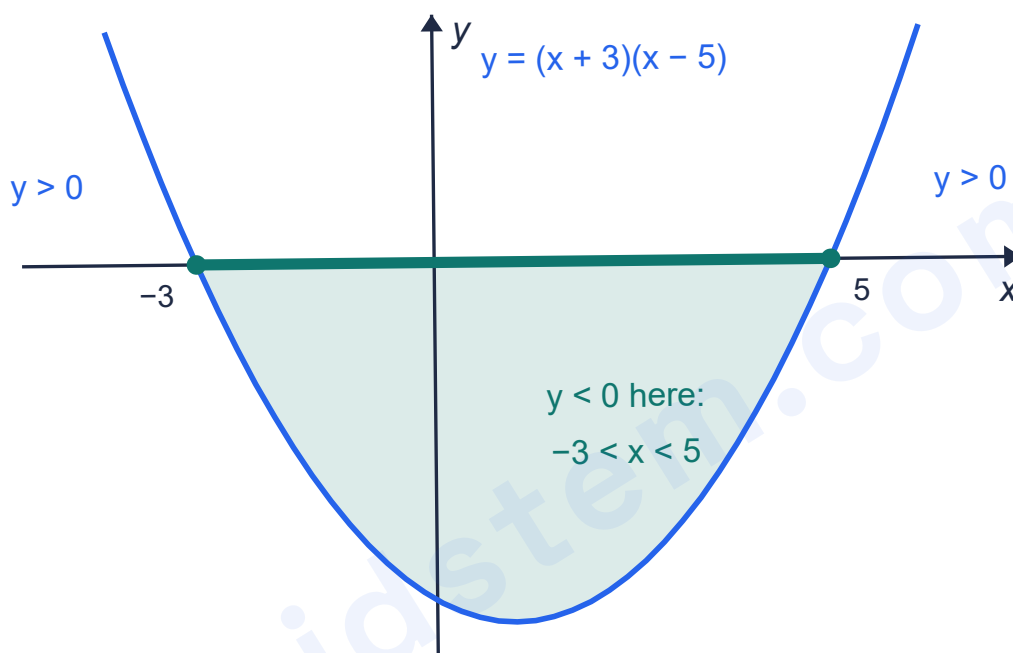


Fig. 1: $y = x^2 - 2x - 15 = (x + 3)(x - 5)$ is below the x-axis between the roots, so $x^2 - 2x - 15 < 0$ for $-3 < x < 5$

Worked example: Factorising

Solve $6x^2 - x - 2 = 0$.

1. Look for two numbers with product $6 \times (-2) = -12$ and sum -1 : -4 and 3 .
2. $6x^2 - 4x + 3x - 2 = 2x(3x - 2) + (3x - 2) = (3x - 2)(2x + 1)$.
3. Set each factor to zero: $3x - 2 = 0$ or $2x + 1 = 0$.

Answer: $x = \frac{2}{3}$ or $x = -\frac{1}{2}$

Zero on one side, roots, sketch, read.

- ✓ **Rearrange** so one side is zero; never divide by x .
- ✓ **Find the critical values** by factorising, completing the square or the formula.
- ✓ **Sketch** the parabola through them (make $a > 0$ by multiplying by -1 and reversing the sign).
- ✓ **Read**: < 0 between the roots, > 0 outside, using $\leq/\geq$ to include the ends.
- ✓ **Write** an outside answer as two pieces joined by 'or'.

Watch out!

Never divide both sides by x :

$x^2 = 4x$ becomes $x = 4$ and the root $x = 0$ is lost. Rearrange to $x^2 - 4x = 0$, so $x(x - 4) = 0$ and $x = 0$ or $x = 4$.

$x^2 > 9$ does NOT mean $x > \pm 3$.

The curve $y = x^2 - 9$ is above the axis outside -3 and 3 , so $x < -3$ or $x > 3$.

An 'outside' answer is two separate pieces joined by 'or'. Writing $6 < k < -2$ is meaningless (no number is both bigger than 6 and smaller than -2) and loses the mark.

4 Simultaneous equations: one linear, one quadratic

Syllabus 1.1

Rearrange the LINEAR equation to make x or y the subject, substitute into the quadratic, solve the resulting quadratic, then substitute each root back into the linear equation. Solutions come in pairs (x, y) : one pair for each point of intersection.

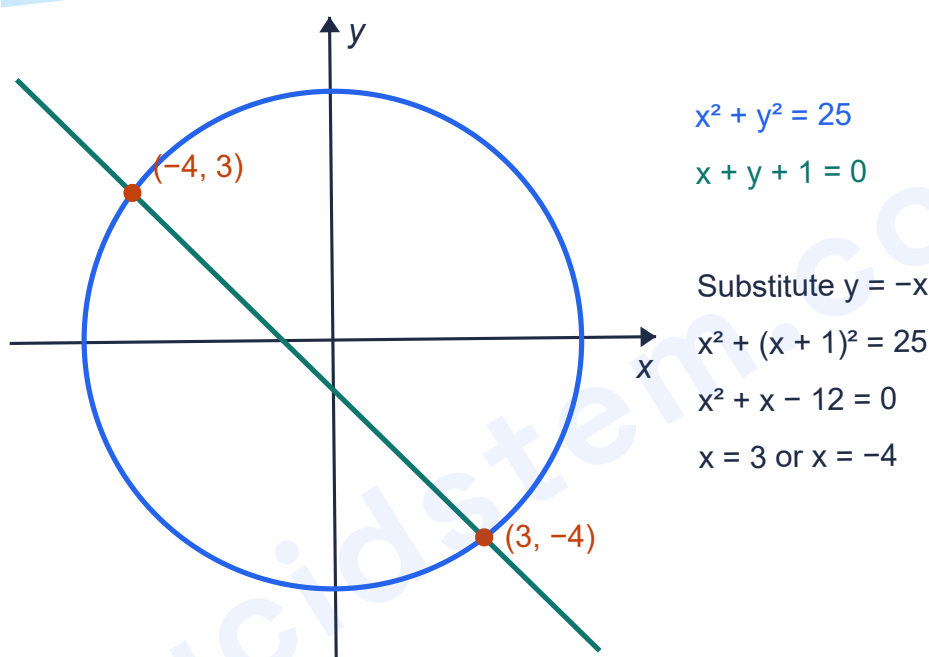


Fig. 1: the line $x + y + 1 = 0$ cuts the circle $x^2 + y^2 = 25$ at $(3, -4)$ and $(-4, 3)$; the two roots of the substituted quadratic are the x -coordinates of these points

Worked example: Line and parabola

Solve the simultaneous equations $y = x + 2$ and $y = x^2 - 4$.

1. Both expressions equal y , so $x^2 - 4 = x + 2$.
2. Rearrange: $x^2 - x - 6 = 0$, so $(x - 3)(x + 2) = 0$.
3. $x = 3$: $y = 3 + 2 = 5$. $x = -2$: $y = -2 + 2 = 0$.

Answer: $x = 3$, $y = 5$ or $x = -2$, $y = 0$

One letter out, one quadratic in.

- ✓ **Rearrange** the linear equation for the letter with the simplest coefficient.
- ✓ **Substitute** everywhere that letter appears in the quadratic, including xy terms.
- ✓ **Expand** squares fully, collect to $ax^2 + bx + c = 0$ and solve.
- ✓ **Back-substitute** each root into the LINEAR equation to find its partner.
- ✓ **Pair** the answers as coordinates; a repeated root means the line is a tangent.

Watch out!

$(-x - 1)^2 = x^2 + 2x + 1$, not $x^2 + 1$. Write the square as a product and expand it fully before collecting terms.

Find each partner from the LINEAR equation, not the quadratic. Substituting $y = 2$ into $x^2 + 3y^2 = 28$ gives $x = \pm 4$, and $(-4, 2)$ is not a solution of $x + 3y = 10$.

Give the answers as matched pairs. Writing ' $x = 3$ or -2 , $y = 5$ or 0 ' does not say which y goes with which x , and examiners withhold the final mark.

5 Equations quadratic in a function of x

Syllabus 1.1

If an equation has the shape $a[f(x)]^2 + bf(x) + c = 0$, put $u = f(x)$, solve $au^2 + bu + c = 0$, then solve $f(x) = u$ for each root. Reject any value that $f(x)$ can never take, such as a negative x^2 or a negative $\sqrt{x}$.

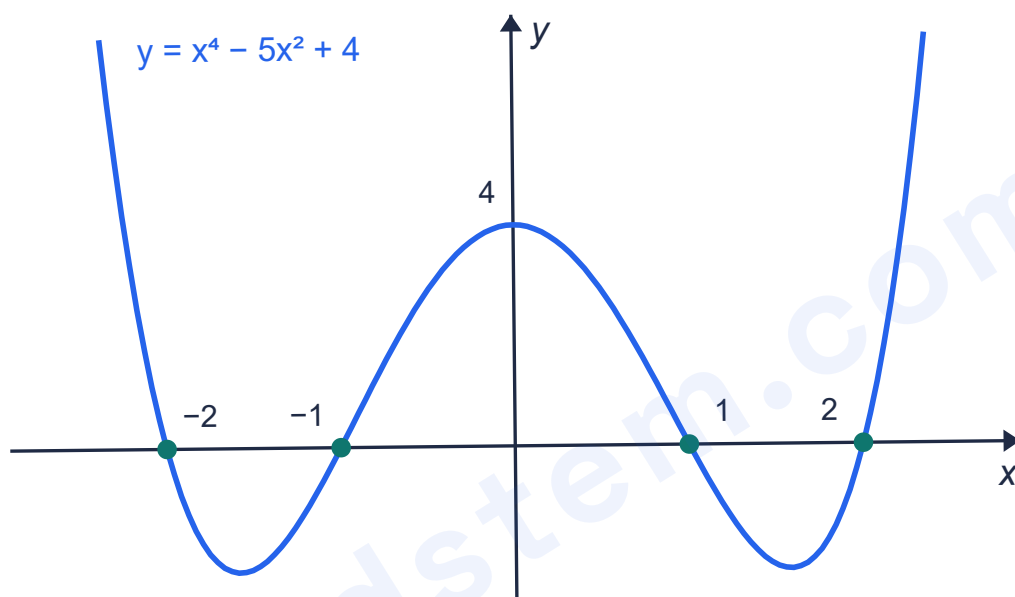


Fig. 1: $y = x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4)$ meets the x -axis four times, at $x = \pm 2$ and $x = \pm 1$

Worked example: Four roots

Solve $x^4 - 5x^2 + 4 = 0$.

1. Let $u = x^2$: $u^2 - 5u + 4 = 0$.
2. $(u - 1)(u - 4) = 0$, so $u = 1$ or $u = 4$.
3. Undo the substitution: $x^2 = 1$ gives $x = \pm 1$; $x^2 = 4$ gives $x = \pm 2$.

Answer: $x = -2, -1, 1, 2$ (the four crossings in Fig. 1)

Find u , solve for u , then find x .

- ✓ **Spot** two terms where one power is double the other.
- ✓ **Substitute** u for the smaller power (or the function) and solve the quadratic in u .
- ✓ **Reject** values the function can never take: negative x^2 , negative $\sqrt{x}$, $\sin x$ outside $[-1, 1]$.
- ✓ **Undo** the substitution for each remaining value, including both signs for $x^2 = k$.
- ✓ **Check** in the original equation when a square root is involved.

Watch out!

Stopping at $u = 1$ or $u = 4$ answers the wrong question. The equation was in x , so convert back, and remember that $x^2 = 4$ has TWO solutions, ± 2 .

Check answers in the ORIGINAL equation. For $6x - \sqrt{x} - 2 = 0$, $x = \frac{1}{4}$ (from $u = -\frac{1}{2}$) gives $1.5 - 0.5 - 2 = -1 \neq 0$, which confirms it must be rejected.

Do not reject a negative value of a cube root: $x^{\frac{1}{3}} = -4$ is fine and gives $x = -64$. Only reject values the function can never take (negative $\sqrt{x}$ or x^2 , $\sin x$ outside -1 to 1 , negative 2^x).