

Series

A-Level Maths 9709 · Topic 1.6 revision notes

Everything from the 3 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. The binomial expansion

2. Arithmetic progressions

3. Geometric progressions and sum to infinity

thelucidstem.com

1 The binomial expansion

Syllabus 1.6

$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. The general term is $\binom{n}{r} a^{n-r} b^r$
: the powers of a and b always add up to n .



Fig. 1: Pascal's triangle. Each entry is the sum of the two above it ($4 + 6 = 10$). Row $n = 5$ gives the coefficients of $(a + b)^5$, and its entries are ${}^5C_0, {}^5C_1, \dots, {}^5C_5$

Worked example: Coefficients without a calculator

Find the values of $\binom{6}{2}$ and $\binom{7}{3}$.

- $\binom{6}{2} = \frac{6!}{2!4!} = \frac{6 \times 5}{2 \times 1} = 15$ (the $4!$ cancels).
- $\binom{7}{3} = \frac{7!}{3!4!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = \frac{210}{6} = 35$.

Answer: $\binom{6}{2} = 15$, $\binom{7}{3} = 35$

General term, then choose r .

- ✓ **Coefficients:** $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ or Pascal's triangle.
- ✓ **Powers:** first term falls from n , second rises from 0; they add to n .
- ✓ **Brackets:** $(-3x)^2 = 9x^2$; keep the sign and the number inside the power.
- ✓ **One term:** simplify the power of x in the general term, then solve for r .
- ✓ **Products:** list every pair that makes the power you want, and add.

Watch out!

$\binom{5}{2}$ is not $\frac{5!}{2!} = 60$. Both factorials go underneath: $\frac{5!}{2!3!} = 10$.

Bracket the whole second term before squaring: $(-3x)^2 = 9x^2$, not $-3x^2$ and not $-9x^2$.

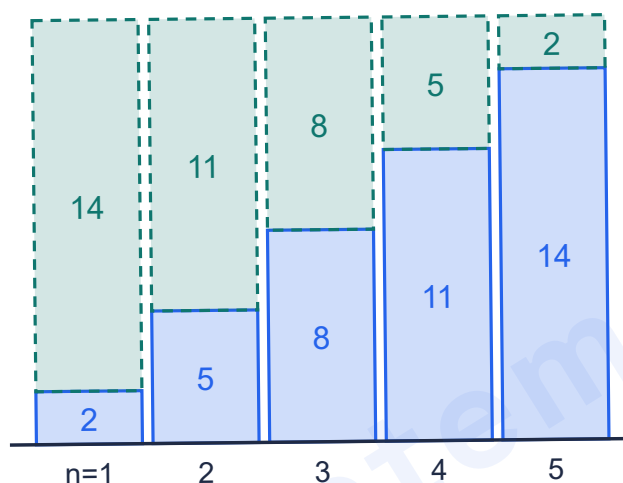
The x^3 term is the **FOURTH** term ($r = 3$), because counting starts at $r = 0$. Using $\binom{6}{4}$ or 3^4 here is the most common slip.

Collect ALL the pairs. In $(1 + 2x)(1 - x)^6$ the answer is not just 15: the $2x$ multiplies the x term to make a second x^2 term.

2 Arithmetic progressions

Syllabus 1.6

An AP with first term a and common difference d has n th term $u_n = a + (n - 1)d$ and sum of the first n terms $S_n = \frac{n}{2} [2a + (n - 1)d] = \frac{n}{2}(a + l)$, where l is the last term. Three numbers a, b, c are in AP when $2b = a + c$.



each column
= $a + l = 16$

$$2S = 5 \times 16$$

$$S = 40$$

Fig. 1: the AP 2, 5, 8, 11, 14 (solid) and the same terms reversed (dashed) fit together into a rectangle 5 columns wide and $a + l = 16$ high, so twice the sum is 5×16 and the sum is 40

Worked example: A term, and which term

For the AP 7, 11, 15, ... find the 20th term, and find which term is equal to 203.

1. $a = 7, d = 4. u_{20} = 7 + 19 \times 4 = 83.$

2. Set $7 + (n - 1) \times 4 = 203$; $(n - 1) \times 4 = 196$, so $n - 1 = 49$.

Answer: $u_{20} = 83$; 203 is the 50th term

Find a and d , then the right formula.

- ✓ **Identify:** a is the first term, d the constant step (it can be negative).
- ✓ **One term:** $u_n = a + (n - 1)d$.
- ✓ **Sum:** $S_n = \frac{n}{2} [2a + (n - 1)d]$ or $\frac{n}{2}(a + l)$.
- ✓ **Unknowns:** two facts give two simultaneous equations in a and d ; three terms in AP give $2b = a + c$.
- ✓ **Inequalities:** solve the quadratic, then take the next whole number and check both sides.

Watch out!

The 20th term is $a + 19d$, not $a + 20d$: the first term already counts as one of the twenty.

Counting terms between two numbers: $\frac{297 - 102}{3} = 65$
counts the gaps; there is one more term than gaps, so $n = 66$.

Read whether the question asks for one term or the total. 'How much in the 24th month' is u_{24} ; 'how much in the first 24 months' is S_{24} .

3 Geometric progressions and sum to infinity

Syllabus 1.6

A GP with first term a and common ratio r has $u_n = ar^{n-1}$ and $S_n = \frac{a(1-r^n)}{1-r}$ ($r \neq 1$). It converges when $|r| < 1$, and then $S_\infty = \frac{a}{1-r}$. Three numbers a, b, c are in GP when $b^2 = ac$.

partial sums of $6 + 3 + 1.5 + 0.75 + \dots$

$S_\infty = 12$

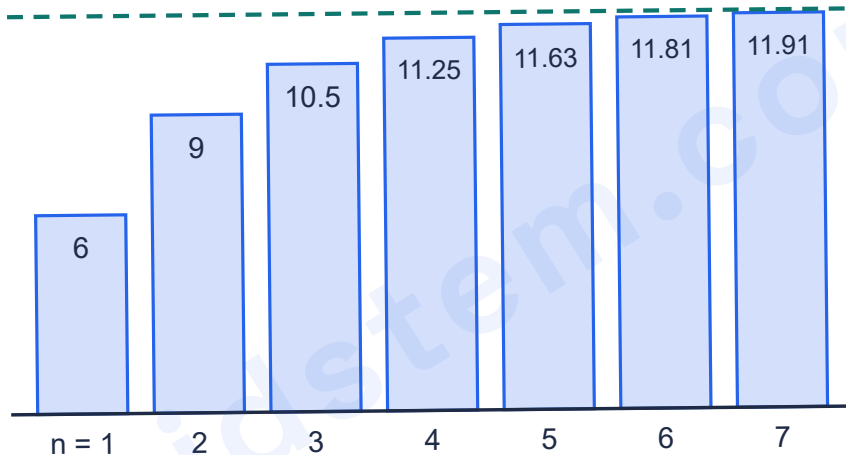


Fig. 1: the partial sums of the GP $6 + 3 + 1.5 + \dots$ ($a = 6, r = 0.5$) creep up towards, but never pass, the sum to infinity $6 \div (1 - 0.5) = 12$

Worked example: A term

Find the 10th term of the GP $3, 6, 12, \dots$

1. $a = 3, r = \frac{6}{3} = 2$.
2. $u_{10} = 3 \times 2^9 = 3 \times 512$.

Answer: 1536

Find a and r , check $|r|$,
then the right formula.

- ✓ **Ratio:** $r = \frac{u_{n+1}}{u_n}$; two given terms:
divide to eliminate a .
- ✓ **One term:** $u_n = ar^{n-1}$.
- ✓ **Sum:** $S_n = \frac{a(1-r^n)}{1-r}$, or
 $\frac{a(r^n-1)}{r-1}$ when $r > 1$.
- ✓ **Sum to infinity:** only if $|r| < 1$; then
 $S_\infty = \frac{a}{1-r}$.
- ✓ **Two progressions:** write the shared
terms with a, d or a, r and use $2b =$
 $a + c$ or $b^2 = ac$.

Watch out!

Dividing ar^5 by ar^2 gives r^3 , not r^4 : subtract the powers, $5 - 2 = 3$.

With a negative ratio, bracket it:

$(-\frac{1}{3})^6 = +\frac{1}{729}$, and $1 - r = 1 - (-\frac{1}{3}) = \frac{4}{3}$, not $\frac{2}{3}$.

S_∞ only exists when $|r| < 1$. A GP such as $2 - 4 + 8 - \dots$ has $r = -2$, so $\frac{a}{1-r} = \frac{2}{3}$ is meaningless.

When an equation like $d^2 = 5d$ appears, do not divide by d blindly: say $d \neq 0$ (given), so $d = 5$. If $d = 0$ were allowed it would also be a solution.