

Trigonometry

A-Level Maths 9709 · Topic 1.5 revision notes

Everything from the 3 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Graphs of trigonometric functions

2. Exact values and the two identities

3. Solving trigonometric equations

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1 Graphs of trigonometric functions

Syllabus 1.5

$y = \sin x$ and $y = \cos x$ have period 2π (360°) and range $-1 \leq y \leq 1$; $y = \tan x$ has period π (180°) and asymptotes at $x = \frac{\pi}{2} + n\pi$. For $y = a \sin bx + c$: amplitude $|a|$, period $\frac{2\pi}{b}$, and the centre line moves to $y = c$.

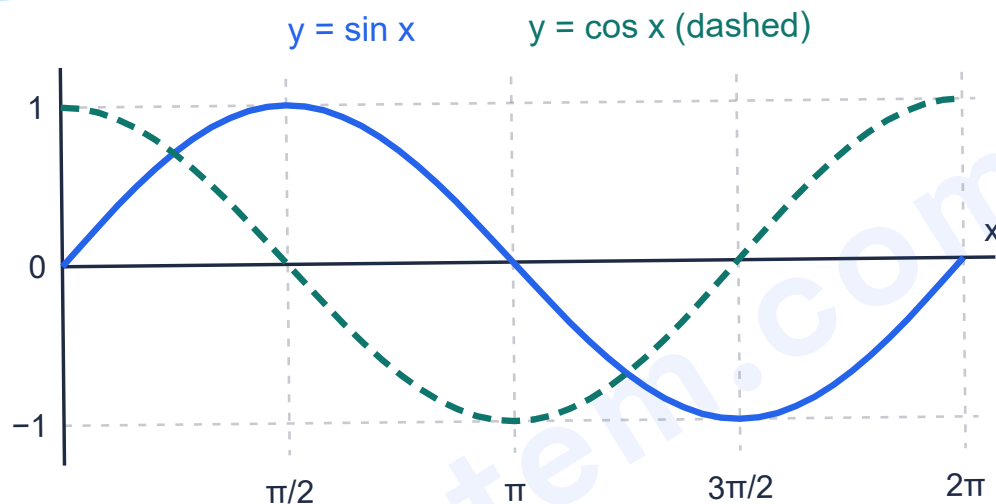


Fig. 1: $y = \sin x$ (solid) and $y = \cos x$ (dashed) for $0 \leq x \leq 2\pi$. The cosine graph is the sine graph translated $\pi/2$ to the left

Worked example: Amplitude and period

State the amplitude and period of $y = 3 \sin 2x$, and its maximum point for $0 \leq x \leq \pi$.

1. The 3 stretches vertically: amplitude 3, so the range is $-3 \leq y \leq 3$.
2. The 2 squeezes horizontally: period $\frac{2\pi}{2} = \pi$.
3. $\sin 2x = 1$ when $2x = \frac{\pi}{2}$, so $x = \frac{\pi}{4}$.

Answer: Amplitude 3, period π , maximum at $(\frac{\pi}{4}, 3)$

Basic shape first, then one transformation at a time.

- ✓ **Shape:** sine starts at 0 going up, cosine starts at its maximum, tangent has asymptotes.
- ✓ **Vertical:** a sets the amplitude $|a|$; c moves the centre line to $y = c$.
- ✓ **Horizontal:** period $\frac{2\pi}{b}$ (tan: $\frac{\pi}{b}$); $x + k$ shifts left by k .
- ✓ **Key points:** label max, min, intercepts and asymptotes with exact coordinates.
- ✓ **Counting:** solutions are crossings with a horizontal line inside the interval.

Watch out!

The tangent graph repeats every π , not every 2π , and it has no maximum or minimum. Draw its asymptotes as dashed vertical lines and make the branches approach them without touching.

$y = \sin(x + k)$ moves the graph LEFT by k , not right; and in $y = \sin 2x$ the period halves, it does not double.

Watch the ends of the interval.

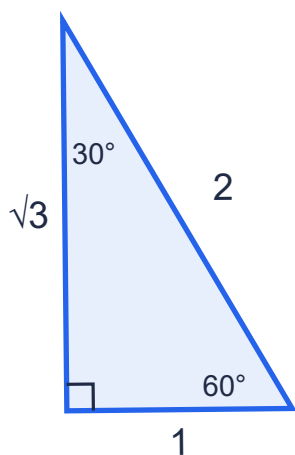
With $0 \leq x \leq \pi$ the end points are included, so $k = 2$ gives two solutions ($x = 0$ and $x = \pi$); with $0 < x < \pi$ it would give none.

2 Exact values and the two identities

Syllabus 1.5

Learn $\sin 30^\circ = \frac{1}{2}$, $\sin 45^\circ = \frac{\sqrt{2}}{2}$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$ (cosine runs the same list backwards) and get every related angle from its sign in that quadrant. For any θ : $\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$ and $\sin^2 \theta + \cos^2 \theta \equiv 1$.

half an equilateral triangle



half a square

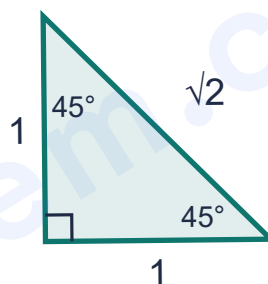


Fig. 1: half of an equilateral triangle of side 2 gives the 30° and 60° ratios; half of a unit square gives the 45° ratios. The missing sides come from Pythagoras

Worked example: Combining exact values

Without a calculator, find the exact value of $2 \sin 60^\circ \cos 30^\circ + \tan^2 30^\circ$.

1. $\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$, so $2 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = 2 \times \frac{3}{4} = \frac{3}{2}$.

2. $\tan 30^\circ = \frac{1}{\sqrt{3}}$, so $\tan^2 30^\circ = \frac{1}{3}$. ($\tan^2 \theta$ means $(\tan \theta)^2$.)

3. $\frac{3}{2} + \frac{1}{3} = \frac{9+2}{6}$.

Answer: $\frac{11}{6}$

Acute angle, quadrant sign, then the identities.

- ✓ **Exact values:** rebuild them from the two special triangles.
- ✓ **Related angles:** acute angle to the x -axis, then the sign of that quadrant.
- ✓ **One ratio to another:** $\sin^2 \theta + \cos^2 \theta = 1$, then choose the sign from the quadrant.
- ✓ **Proofs:** start from one side, combine fractions, replace $1 - \cos^2 \theta$ by $\sin^2 \theta$ (or the reverse), finish on the other side.
- ✓ **Exact answers:** rationalise surd denominators when asked.

Watch out!

$\tan^2 30^\circ$ is $(\tan 30^\circ)^2 = \frac{1}{3}$, not $\tan(900^\circ)$ and not $\tan 30^\circ$ doubled.

Find the acute angle to the x -axis, not to the y -axis. 150° is 30° from the axis ($180^\circ - 150^\circ$), not 60° ($150^\circ - 90^\circ$).

After $\cos^2 \theta = \frac{144}{169}$, the square root has two signs. The quadrant decides which one; dropping the minus sign for an obtuse angle loses the marks.

3 Solving trigonometric equations

Syllabus 1.5

Find the principal value α with $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$. Then $\sin x = k$ also has $180^\circ - \alpha$; $\cos x = k$ also has $360^\circ - \alpha$; $\tan x = k$ also has $\alpha + 180^\circ$. Add or subtract whole periods to stay inside the interval, and for $\sin 2x$ widen the interval for $2x$ first.

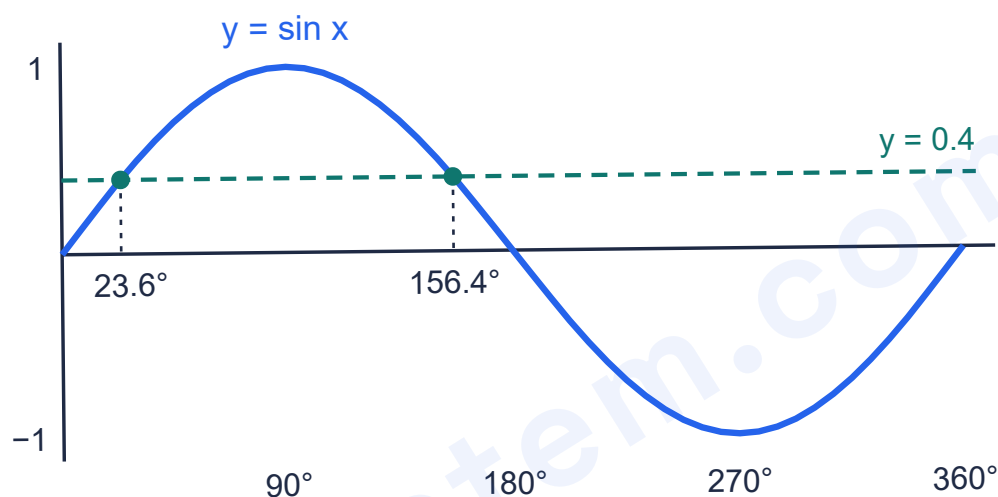


Fig. 1: $\sin x = 0.4$ for $0^\circ \leq x \leq 360^\circ$. The calculator gives 23.6° ; the symmetry of the sine curve about $x = 90^\circ$ gives the second solution $180^\circ - 23.6^\circ = 156.4^\circ$

Worked example: Exact principal values

Find the exact values of (a) $\sin^{-1}(-\frac{1}{2})$ (b) $\cos^{-1}(-\frac{1}{2})$ (c) $\tan^{-1}(-\sqrt{3})$.

- (a) $\sin \frac{\pi}{6} = \frac{1}{2}$; the principal value must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so take $-\frac{\pi}{6}$.
- (b) The range of $\cos^{-1}$ is 0 to π ; a negative cosine needs the second quadrant:
 $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$.
- (c) $\tan \frac{\pi}{3} = \sqrt{3}$; the principal value of $\tan^{-1}$ of a negative number is negative.

Answer: (a) $-\frac{\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $-\frac{\pi}{3}$

Principal value, symmetry, period, interval.

- ✓ **One function:** use $\tan x = \frac{\sin x}{\cos x}$ or $\sin^2 x + \cos^2 x = 1$ so only one trig function is left; factorise rather than cancel.
- ✓ **Bracket interval:** for $\sin 2x$ or $\cos(x + k)$, rewrite the interval for the whole bracket.
- ✓ **Principal value:** $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ on the calculator, in the right mode.
- ✓ **Second value:** $180^\circ - \alpha$ (sin), $360^\circ - \alpha$ (cos), $\alpha + 180^\circ$ (tan).
- ✓ **Period:** add or subtract 360° (tan: 180°) until you leave the interval, then convert back to x .

Watch out!

$\sin^{-1} x$ is NOT $\frac{1}{\sin x}$. The -1 means inverse function, just as f^{-1} does; $(\sin x)^{-1}$ is a different thing.

A negative principal value is not automatically wrong, and it is not automatically an answer. If the interval is 0° to 360° and the calculator gives -63.4° , add 180° (or 360°) to bring it into range.

Halving too early loses solutions. If you solve $\sin 2x = -\frac{1}{2}$ only for $-\pi < 2x < \pi$ you find two answers instead of four.

Never cancel a trig function that could be zero. In $\sin x = 2 \sin x \cos x$, dividing by $\sin x$ loses the solutions of $\sin x = 0$. Instead factorise: $\sin x(1 - 2 \cos x) = 0$.