

Vectors

A-Level Maths 9709 · Topic 3.7 revision notes

Everything from the 4 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

1. Vectors in two and three dimensions

3. Parallel, intersecting and skew lines

2. The vector equation of a line

4. The scalar product

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1 Vectors in two and three dimensions

Syllabus 3.7

A vector in space is $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, with magnitude $\sqrt{x^2 + y^2 + z^2}$. The displacement from A to B is $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$: end minus start.

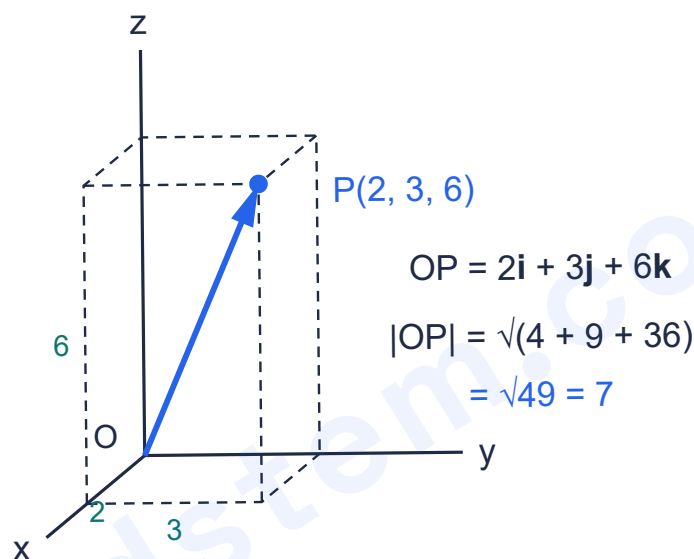


Fig. 1: $P(2, 3, 6)$ sits at a corner of a 2 by 3 by 6 box; its position vector is $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ and its distance from O is the length of the box diagonal, 7

Worked example: Magnitude and unit vector

Given $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$, find $|\mathbf{a}|$ and the unit vector in the direction of $\mathbf{a}$.

1. Square each component: $2^2 = 4$, $(-3)^2 = 9$, $6^2 = 36$. A negative component still gives a positive square.
2. $|\mathbf{a}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$.
3. Divide the vector by its magnitude: $\hat{\mathbf{a}} = \frac{1}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$.
4. Check: $\left(\frac{2}{7}\right)^2 + \left(\frac{-3}{7}\right)^2 + \left(\frac{6}{7}\right)^2 = \frac{49}{49} = 1$.

Answer: $|\mathbf{a}| = 7$; unit vector $\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$.

*End minus start, then
Pythagoras.*

- ✓ **Displacement:** $\vec{AB} = \mathbf{b} - \mathbf{a}$.
- ✓ **Magnitude:** $\sqrt{x^2 + y^2 + z^2}$; unit vector = vector $\div$ magnitude.
- ✓ **Parallel:** one vector is a single multiple λ of the other.
- ✓ **Collinear:** $\vec{AB}$ parallel to $\vec{BC}$ with B shared.
- ✓ **New points:** known point + (fraction of a displacement) or + (distance $\times$ unit vector).

Watch out!

$\vec{AB}$ is $\mathbf{b} - \mathbf{a}$, not $\mathbf{a} - \mathbf{b}$.

Getting it backwards gives $\vec{BA}$: the same length but the opposite direction, which ruins every later step that uses the direction.

Parallel vectors need **ONE** common factor for every component.

$(2, 4, 6)$ and $(1, 2, 4)$ look alike, but the factors are 2, 2, 1.5, so they are not parallel.

The midpoint is half the **SUM** of the position vectors, $\frac{1}{2}(\mathbf{a} + \mathbf{b})$. Half the difference, $\frac{1}{2}(\mathbf{b} - \mathbf{a})$, is the step from A to the midpoint, not the midpoint itself.

2 The vector equation of a line

Syllabus 3.7

The line through the point with position vector $\mathbf{a}$, in the direction of $\mathbf{b}$, is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. Each value of the parameter t gives one point on the line, and every point on the line comes from exactly one t .

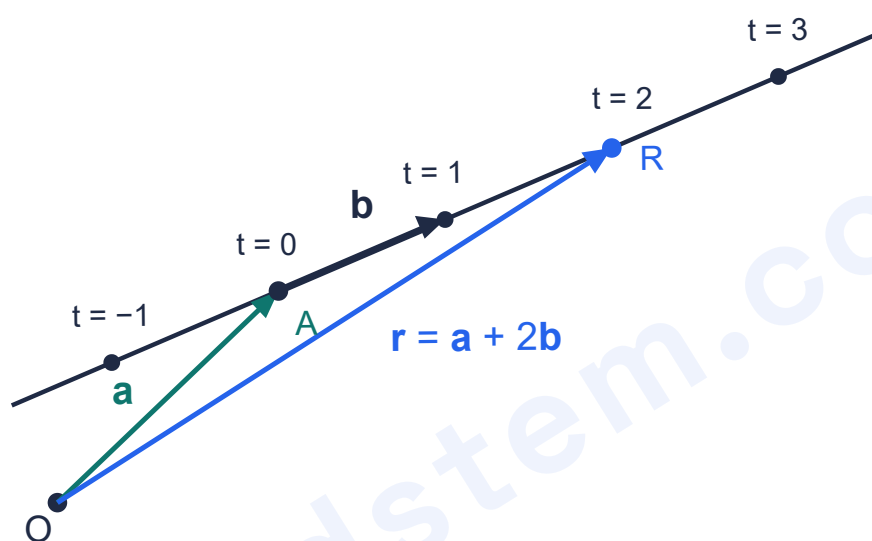


Fig. 1: start at A (position vector $\mathbf{a}$) and step along the direction $\mathbf{b}$; the parameter t counts the steps, so $t = 2$ reaches the point R with $\mathbf{r} = \mathbf{a} + 2\mathbf{b}$, and negative t goes backwards past A

Worked example: Points from parameters

The line l passes through $A(1, -2, 3)$ and is parallel to $2\mathbf{i} + \mathbf{j} - \mathbf{k}$. Write down a vector equation of l , and find the point on l where $z = 0$.

1. Point plus multiple of direction: $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

2. A general point is $(1 + 2t, -2 + t, 3 - t)$.

3. $z = 0$: $3 - t = 0$, so $t = 3$.

4. Substitute $t = 3$: $(1 + 6, -2 + 3, 0)$.

Answer: $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + t(2\mathbf{i} + \mathbf{j} - \mathbf{k})$; the point is $(7, 1, 0)$.

A point, a direction, a parameter.

- ✓ **Form:** $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, with $\mathbf{a}$ a point on the line and $\mathbf{b}$ a direction.
- ✓ **Two points:** $\mathbf{b} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$; either point can be $\mathbf{a}$.
- ✓ **Point on line?** Find t from one component, then check it in the other two.
- ✓ **Same line?** Parallel directions plus one shared point.
- ✓ **Distance:** a parameter change of t moves $|t| |\mathbf{b}|$.

Watch out!

Finding t from ONE component proves nothing. A point lies on the line only if the SAME t works in all three components; always check the remaining two.

The direction is $\mathbf{b} - \mathbf{a}$, never the position vector of the second point. Writing $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ with $\mathbf{b} = \overrightarrow{OB}$ gives a line through A parallel to OB , which usually misses B completely.

The parameter is not a distance unless $\mathbf{b}$ is a unit vector. With $\mathbf{b} = (4, 2, -2)$, $t = 5$ means $5\sqrt{24}$ units, not 5 units.

3 Parallel, intersecting and skew lines

Syllabus 3.7

First compare the directions: if one is a multiple of the other, the lines are parallel (or the same line). If not, set the two general points equal, solve two component equations for the two parameters, and test the third: it holds when the lines intersect and fails when they are skew.

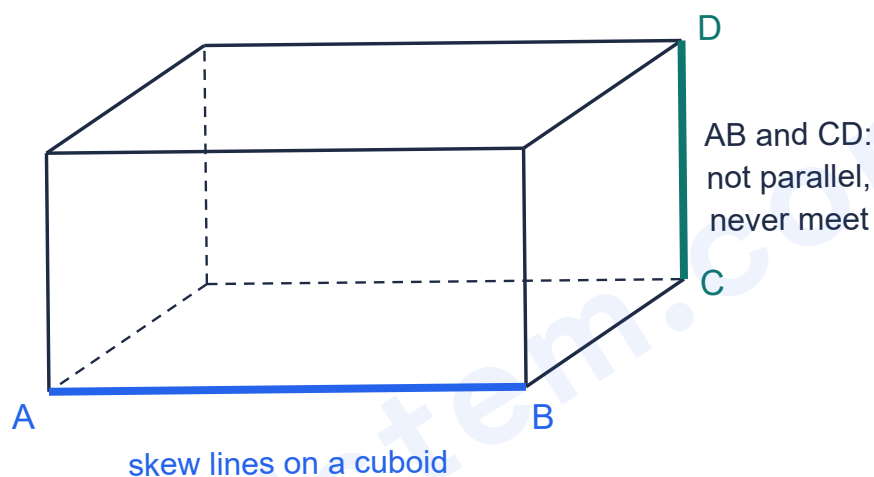


Fig. 1: on a cuboid the edge AB runs along the length and the edge CD runs up the back; they point in different directions yet never meet, so they are skew

Worked example: Parallel or the same line?

$l_1: \mathbf{r} = (1, 2, 3) + s(1, -1, 2)$ and $l_4: \mathbf{r} = (0, 4, 1) + u(-2, 2, -4)$. Show that l_1 and l_4 are parallel and distinct.

1. $(-2, 2, -4) = -2(1, -1, 2)$, so the directions are parallel.
2. Test whether $(0, 4, 1)$ lies on l_1 : $1 + s = 0$ gives $s = -1$.
3. Then $y = 2 - (-1) = 3 \neq 4$, so the point is not on l_1 .

Answer: Parallel directions but no common point: parallel, distinct lines.

Directions first, then three equations.

- ✓ **Parallel?** Is one direction a multiple of the other? If so, test one point to decide parallel or the same line.
- ✓ **Otherwise** equate general points, with different parameters for the two lines.
- ✓ **Solve** two of the three component equations for the two parameters.
- ✓ **Test** the third equation: true means intersecting, false means skew.
- ✓ **Point:** substitute each parameter into its OWN line, and check both agree.

Watch out!

In two dimensions, non-parallel lines must meet, so students skip the check. In three dimensions non-parallel is not enough: you must show the lines share a point before calling them intersecting.

Using the same letter for both parameters (say t and t) forces the two points to have equal parameters, which they almost never do. You then wrongly conclude 'skew'. Always use s and t , or λ and μ .

Intersecting PATHS do not mean a collision. The parameters are separate; a collision needs both objects at the point at the same time, which means equal times, not just equal positions.

4 The scalar product

Syllabus 3.7

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3$. So $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$, and non-zero vectors are perpendicular exactly when $\mathbf{a} \cdot \mathbf{b} = 0$.

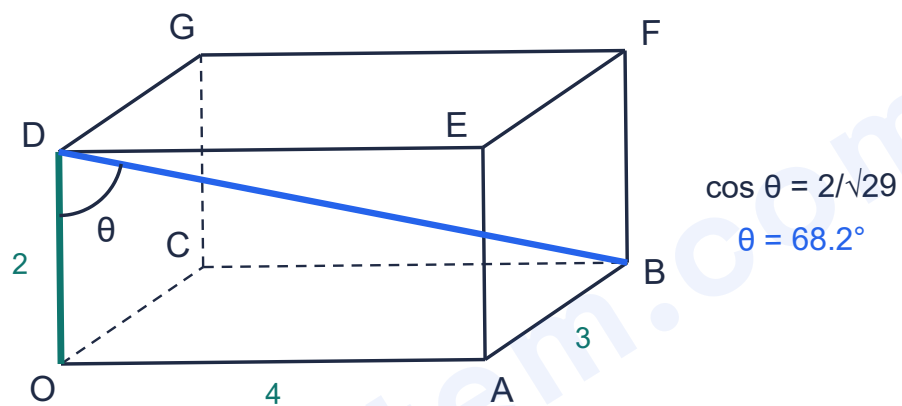


Fig. 1: cuboid $OABCDEFG$ with $OA = 4$, $OC = 3$ and $OD = 2$; the angle ODB between the edge DO and the diagonal DB is 68.2°

Worked example: Angle between two vectors

Find the angle between $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$.

1. $\mathbf{a} \cdot \mathbf{b} = 2(1) + (-1)(4) + 2(-8) = 2 - 4 - 16 = -18$.
2. $|\mathbf{a}| = \sqrt{4 + 1 + 4} = 3$ and $|\mathbf{b}| = \sqrt{1 + 16 + 64} = 9$.
3. $\cos \theta = \frac{-18}{3 \times 9} = -\frac{2}{3}$.
4. $\theta = \cos^{-1}\left(-\frac{2}{3}\right) = 131.8^\circ$. Obtuse, as the negative product predicted.

Answer: $\theta = 131.8^\circ$ (1 d.p.)

Components for the number, cosine for the angle.

- ✓ **Calculate:** $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- ✓ **Angle:** $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$, with both vectors pointing away from the vertex.
- ✓ **Lines:** use the direction vectors and give the acute angle.
- ✓ **Perpendicular:** set the scalar product equal to zero.
- ✓ **Foot of perpendicular:** general point N , then $\overrightarrow{PN} \cdot \mathbf{b} = 0$.

Watch out!

The angle in the formula is between vectors pointing away from the same point. For the angle ABC in a triangle use $\overrightarrow{BA}$ and $\overrightarrow{BC}$; using $\overrightarrow{AB}$ and $\overrightarrow{BC}$ gives 180° minus the angle you want.

Use the DIRECTION vectors of the lines, not the position vectors of the points on them. The position vectors depend on where the origin is; the angle between the lines does not.

Dot $\overrightarrow{PN}$ with the DIRECTION of the line, not with the position vector of N or of P . And remember to finish: the question usually wants N or PN , not just the parameter.