

Algebra and graphs

IGCSE Maths 0580 · Topic 2 revision notes

Everything from the 22 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

- | | |
|--|---|
| 1. <u>Algebraic notation and substitution</u> | 13. <u>Sequences and the nth term</u> |
| 2. <u>Expanding brackets and simplifying</u> | 14. <u>Quadratic, cubic and exponential sequences</u> |
| 3. <u>Factorising expressions</u> | 15. <u>Direct and inverse proportion</u> |
| 4. <u>Algebraic fractions</u> | 16. <u>Conversion, distance-time and speed-time graphs</u> |
| 5. <u>Indices II: algebraic and fractional indices</u> | 17. <u>Graphs of functions: tables, plotting and tangents</u> |
| 6. <u>Forming and solving linear equations</u> | 18. <u>Solving equations graphically</u> |
| 7. <u>Changing the subject of a formula</u> | 19. <u>Sketching curves: roots, intercepts and turning points</u> |
| 8. <u>Simultaneous equations</u> | 20. <u>Differentiation: gradient functions</u> |
| 9. <u>Solving quadratic equations</u> | 21. <u>Stationary points</u> |
| 10. <u>Completing the square</u> | 22. <u>Functions: notation, inverse and composite</u> |
| 11. <u>Linear inequalities and the number line</u> | |
| 12. <u>Inequalities as regions on a graph</u> | |

1 Algebraic notation and substitution

Syllabus 2.1

A letter stands for a number. To substitute, replace each letter with its value in brackets, then follow the order of operations: brackets, then powers, then multiply and divide, then add and subtract.

Worked example: Words into algebra

Write an expression for (a) a number that is 2 more than n (b) three times n , then subtract 5 (c) add 5 to n , then multiply by 3.

1. (a) 2 more than n means add 2 to n .
2. (b) Three times n is $3n$; subtracting 5 gives $3n - 5$.
3. (c) The adding happens first, so it must go in a bracket before multiplying by 3.

Answer: (a) $n + 2$ (b) $3n - 5$ (c) $3(n + 5)$

Brackets round every value, then the order of operations.

- ✓ **Notation:** $3a = 3 \times a$, $ab = a \times b$, $3a^2 = 3 \times a^2$.
- ✓ **Words to algebra:** if something happens first, it may need a bracket: $3(n + 5)$.
- ✓ **Substitute:** write each value in brackets, especially negatives.
- ✓ **Order:** brackets, powers, multiply/divide, add/subtract.
- ✓ **Check signs:** minus a negative is plus; a negative squared is positive.

Watch out!

$n(n + 1)$ is the product of two consecutive numbers, but not necessarily even ones. Writing $2n$ guarantees the number is even.

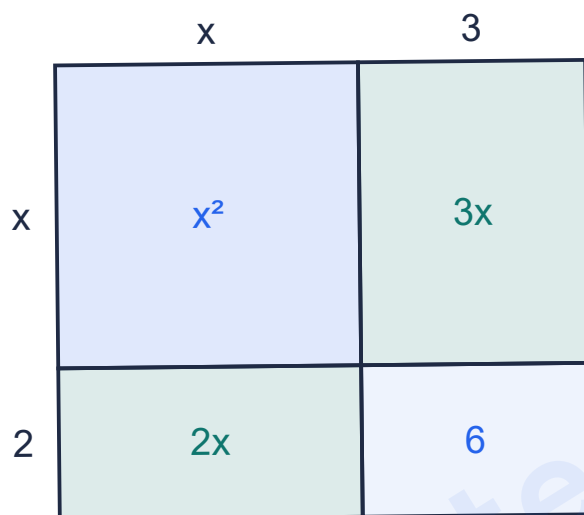
$2x^2$ means $2 \times x^2$, not $(2x)^2$. With $x = -3$: $2x^2 = 18$ but $(2x)^2 = 36$. And a calculator reads -3^2 as -9 : type $(-3)^2$ with the brackets.

Money answers need two decimal places: \$127.5 should be written \$127.50.

2 Expanding brackets and simplifying

Syllabus 2.2

To expand, multiply every term in one bracket by every term in the other, keeping each sign with the term after it. Then collect like terms: only terms with exactly the same letters and powers can be added.



$$(x + 3)(x + 2)$$

$$= x^2 + 3x + 2x + 6$$

$$= x^2 + 5x + 6$$

Fig. 1: the area of a rectangle $(x + 3)$ by $(x + 2)$ splits into four parts, one for each pair of terms, so $(x + 3)(x + 2) = x^2 + 5x + 6$

Worked example: Simplify (Extended)

Simplify $2a^2 + 3ab - 1 + 5a^2 - 9ab + 4$.

1. a^2 terms: $2a^2 + 5a^2 = 7a^2$.

2. ab terms: $3ab - 9ab = -6ab$.

3. Numbers: $-1 + 4 = 3$.

Answer: $7a^2 - 6ab + 3$

Multiply every pair, then collect.

- ✓ **Like terms:** same letters, same powers. Add their numbers only.
- ✓ **One bracket:** the outside term multiplies every inside term, sign included.
- ✓ **Two brackets:** four products, then collect the middle two.
- ✓ **Squares:** write $(a + b)^2$ as $(a + b)(a + b)$.
- ✓ **Three brackets (Extended):** expand two, simplify, then multiply by the third.

Watch out!

$-2(a - 4)$ is $-2a + 8$, not $-2a - 8$. The minus sign outside multiplies the -4 too, and minus times minus is plus.

$(x + 3)^2$ is NOT $x^2 + 9$. Write it as $(x + 3)(x + 3)$ and you get the two middle terms: $x^2 + 6x + 9$.

Simplify the first pair before multiplying by the third bracket. Carrying four unsimplified terms forward doubles the chance of a sign slip.

3 Factorising expressions

Syllabus 2.2

Always take out the highest common factor first, then look inside the bracket for a pattern: a quadratic, a difference of two squares or a perfect square. Factorise fully means nothing left can be factorised again. Check by expanding.

Worked example: Two terms

Factorise fully $9x^2 + 15xy$.

1. HCF of 9 and 15 is 3; both terms contain x ; only one contains y . So the HCF is $3x$.
2. $9x^2 \div 3x = 3x$ and $15xy \div 3x = 5y$.

Answer: $3x(3x + 5y)$

Common factor first, then a pattern.

- ✓ **HCF:** biggest number, and each letter to its lowest power.
- ✓ $x^2 + bx + c$: two numbers that multiply to c and add to b .
- ✓ $ax^2 + bx + c$: numbers that multiply to ac and add to b ; split and group.
- ✓ **Difference of two squares:** $p^2 - q^2 = (p - q)(p + q)$.
- ✓ **Check:** expand your answer and look for anything still common.

Watch out!

$3(3x^2 + 5xy)$ is factorised, but not fully: x is still common inside the bracket. 'Factorise fully' loses a mark for this.

Getting the right numbers with the wrong signs: $(x + 5)(x - 3) = x^2 + 2x - 15$, not $x^2 - 2x - 15$. Expand to check the middle term every time.

$x^2 + 25$ does NOT factorise. The difference of two squares needs a minus sign between two perfect squares.

4 Algebraic fractions

Syllabus 2.3

Only cancel factors, never terms. Factorise the top and bottom completely first; then a bracket that appears on both can be cancelled. To add or subtract, write both fractions over a common denominator.

Worked example: Cancel a common bracket

Simplify $\frac{x^2 - 2x}{x^2 - 5x + 6}$.

1. Top: common factor x , so $x^2 - 2x = x(x - 2)$.
2. Bottom: product 6, sum -5 , so $x^2 - 5x + 6 = (x - 2)(x - 3)$.
3. $\frac{x(x - 2)}{(x - 2)(x - 3)}$: cancel the common factor $(x - 2)$.

Answer: $\frac{x}{x - 3}$

Same rules as number fractions.

- ✓ **Simplify:** factorise top and bottom fully, then cancel common factors.
- ✓ **Multiply:** tops times tops, bottoms times bottoms, cancelling first.
- ✓ **Divide:** flip the second fraction and multiply.
- ✓ **Add/subtract:** common denominator; numerators in brackets.
- ✓ **Finish:** factorise the final numerator to see if anything else cancels.

Watch out!

Cancelling the x^2 terms in $\frac{x^2 + 3x}{x^2 - 9}$ to get $\frac{3x}{-9}$ is wrong: x^2 is a term, not a factor.

Factorise first: the answer is $\frac{x}{x - 3}$.

Flip the **SECOND** fraction only, and only when dividing. Flipping the first, or flipping when multiplying, gives the reciprocal of the right answer.

In $\frac{x + 1}{4} - \frac{x - 3}{6}$ the minus applies to the whole of $x - 3$: $3(x + 1) - 2(x - 3) = 3x + 3 - 2x + 6$. Writing -6 instead of $+6$ is the classic slip.

5 Indices II: algebraic and fractional indices

Syllabus 2.4

Multiply: add the powers. Divide: subtract the powers. Power of a power: multiply the powers.

$x^0 = 1$, $x^{-n} = \frac{1}{x^n}$, and (Extended) $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$. Numbers in front are multiplied or divided as normal.

Worked example: Power of a product

Simplify $(5x^3)^2$.

1. Everything inside the bracket is squared: $5^2 \times (x^3)^2$.
2. $5^2 = 25$ and $(x^3)^2 = x^6$.

Answer: $25x^6$

Numbers as normal, powers by the laws.

- ✓ **Multiply:** add the powers; multiply the numbers in front.
- ✓ **Divide:** subtract the powers; divide the numbers in front.
- ✓ **Bracket to a power:** every factor inside gets the power.
- ✓ **Negative and zero:** $x^{-n} = \frac{1}{x^n}$, $x^0 = 1$.
- ✓ **Fractional (Extended):** bottom is the root, top is the power.
- ✓ **Equations:** same base on both sides, then equate the powers.

Watch out!

$(5x^3)^2$ is not $5x^6$: the 5 is inside the bracket, so it is squared too. And the numbers in front are multiplied, never added: $3x^2 \times 4x^3 = 12x^5$.

$3x^{-2}$ is $\frac{3}{x^2}$, not $\frac{1}{3x^2}$. The power belongs only to the letter it touches.

$8^{\frac{2}{3}}$ is not $8 \times \frac{2}{3}$. The fraction is an instruction: cube root, then square.

$\frac{1}{27}$ is 3^{-3} , not 3^3 . A fraction 'one over' needs a negative power.

6 Forming and solving linear equations

Syllabus 2.5

Whatever you do to one side of an equation, do to the other. Expand brackets, collect the unknown on one side and the numbers on the other, then divide. Check by substituting your answer back into the original equation.

Worked example: Two steps

Solve $3x + 4 = 10$.

1. Subtract 4 from both sides: $3x = 6$.
2. Divide both sides by 3.

Answer: $x = 2$

Expand, collect, divide, check.

- ✓ **Fractions (Extended):** multiply every term by the common denominator first.
- ✓ **Brackets:** expand, watching the sign of the number outside.
- ✓ **Collect:** unknowns on the side where they stay positive, numbers on the other.
- ✓ **Divide:** by the number in front of the unknown.
- ✓ **Check:** substitute into the original equation; answer what was asked.

Watch out!

In $8 - 3(x - 2)$, the -3 multiplies the -2 as well: $-3 \times (-2) = +6$. Writing $8 - 3x - 6$ is the most common slip on this type.

Finding x is often not the end. Read the question again: it may ask for an angle, a length or an age built from x .

Multiplying only the fractions by 6 and forgetting the 1 on the right gives $7x + 4 = 1$. Every term, on both sides, must be multiplied.

7 Changing the subject of a formula

Syllabus 2.5

Undo what has been done to the new subject, in reverse order, doing the same to both sides. If the subject appears twice (Extended), collect both terms on one side, factorise it out, then divide.

Worked example: Two operations

Make t the subject of $v = u + at$.

1. t was multiplied by a , then u was added. Undo the adding first: $v - u = at$.
2. Undo the multiplying: divide both sides by a .

Answer: $t = \frac{v - u}{a}$

Treat it like an equation.

- ✓ **Fractions:** multiply both sides by the denominator first.
- ✓ **Reverse order:** undo the last operation done to the subject first.
- ✓ **Whole sides:** divide, square or root the whole of each side.
- ✓ **Subject twice (Extended):** collect, factorise, divide by the bracket.
- ✓ **Check:** put numbers into the old and new formula.

Watch out!

From $4y + 5 = 3x$, writing $x = \frac{4y}{3} + 5$ divides only one term.

The whole side must be divided:

$$x = \frac{4y + 5}{3}.$$

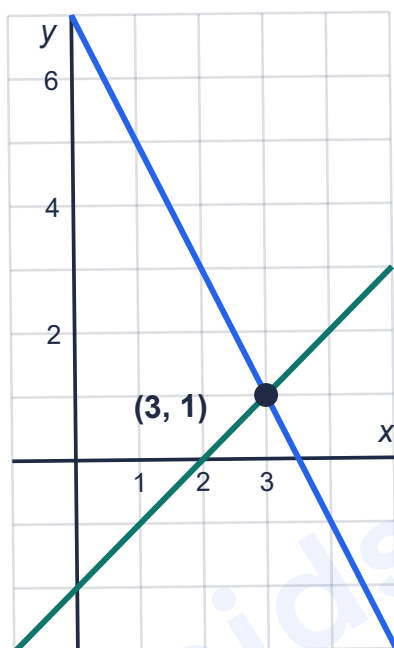
When you square $\frac{T}{2\pi}$, the 2π is squared too: $4\pi^2$, not 2π .

An answer such as $x = \frac{3y + 2 + x}{y}$ still has x on the right, so it is not rearranged. The subject must appear on one side only, exactly once.

8 Simultaneous equations

Syllabus 2.5

Make the coefficients of one unknown the same size, then add the equations if the signs differ or subtract if they are the same. Solve for the other unknown, substitute back, and check both values in both equations. The solution is where the two graphs cross.



— $2x + y = 7$

— $x - y = 2$

Add: $3x = 9$, so $x = 3$

Then $y = 3 - 2 = 1$

The lines cross at $(3, 1)$

Fig. 1: the graphs of $2x + y = 7$ and $x - y = 2$ cross at $(3, 1)$, the one pair of values that makes both equations true

Worked example: Add to eliminate

Solve $2x + y = 7$ and $x - y = 2$.

1. The y terms are $+y$ and $-y$: different signs, so add the equations.
2. $3x = 9$, so $x = 3$.
3. Substitute into $x - y = 2$: $3 - y = 2$, so $y = 1$.
4. Check in the first: $2(3) + 1 = 7$. Correct.

Answer: $x = 3$, $y = 1$

Match, then add or subtract.

- ✓ **Match:** multiply so one unknown has the same coefficient in both.
- ✓ **Signs:** different signs add; same signs subtract.
- ✓ **Substitution:** best when one equation already says $y = \dots$
- ✓ **Back-substitute:** find the second unknown, then check both equations.
- ✓ **Non-linear (Extended):** substitute the linear into the other and solve the quadratic; give pairs.

Watch out!

Same Signs Subtract. When you subtract $2x - y = 5$ from $2x + 3y = 17$, the y terms give $3y - (-y) = 4y$, not $2y$.

Write down what each letter stands for, with units. Many students find the right numbers and then give the child price when the question asked for the adult price.

$(x + 1)^2$ is $x^2 + 2x + 1$, not $x^2 + 1$. And give the answers as pairs: each x goes with its own y .

9 Solving quadratic equations

Syllabus 2.5

Rearrange to $ax^2 + bx + c = 0$ first. If it factorises, set each bracket equal to zero. If not, use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The solutions are where the curve $y = ax^2 + bx + c$ crosses the x -axis.

Worked example: Rearrange, then factorise

Solve $x^2 + 3x = 10$.

1. Make one side zero: $x^2 + 3x - 10 = 0$.
2. Product -10 , sum 3 : 5 and -2 , so $(x + 5)(x - 2) = 0$.
3. $x + 5 = 0$ or $x - 2 = 0$.

Answer: $x = -5$ or $x = 2$

Rearrange to zero, then factorise or use the formula.

- ✓ **Zero:** rearrange to $ax^2 + bx + c = 0$.
- ✓ **Factorise:** if you can, set each bracket to zero.
- ✓ **Formula:** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; work out $b^2 - 4ac$ first.
- ✓ **Form:** exact (surds) or rounded, as asked.
- ✓ **Context:** reject impossible answers and answer the actual question.

Watch out!

Dividing $x^2 = 7x$ by x gives only $x = 7$ and loses $x = 0$. Never divide by the unknown; factorise instead.

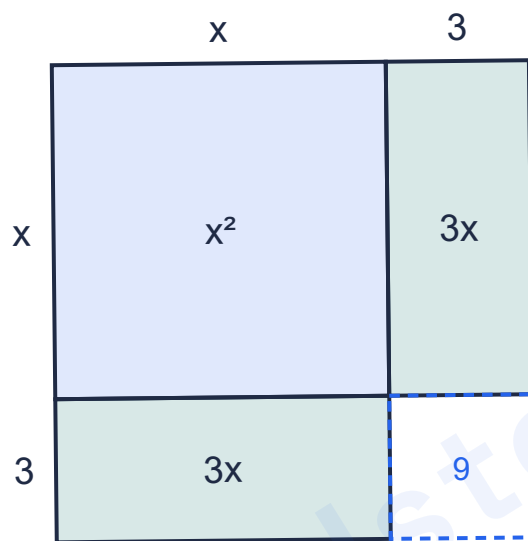
When b is negative, $-b$ is positive: for $b = -3$, $-b = +3$. And $-4ac$ with $c = -4$ becomes $+32$. Writing every value in brackets prevents both slips.

Do not stop at the solutions of the quadratic. Reject impossible values and answer the question that was asked, which may be a perimeter or a length rather than x .

10 Completing the square

Syllabus 2.2

Halve the coefficient of x : $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$. If $a \neq 1$, take a out of the x^2 and x terms first. The graph of $y = (x + p)^2 + q$ has its minimum point at $(-p, q)$.



$$x^2 + 6x$$

$$= (x + 3)^2 - 9$$

half of 6 is 3;
the 3×3 corner
is missing, so take 9

Fig. 1: $x^2 + 6x$ is a square of side x with two strips 3 by x . Moved into an L, they almost make a square of side $(x + 3)$, but a 3 by 3 corner is missing, so $x^2 + 6x = (x + 3)^2 - 9$

Worked example: Positive b

Write $x^2 + 6x + 5$ in the form $(x + p)^2 + q$.

1. Half of 6 is 3, so $x^2 + 6x = (x + 3)^2 - 9$.

2. $x^2 + 6x + 5 = (x + 3)^2 - 9 + 5$.

Answer: $(x + 3)^2 - 4$

Halve b , square it, subtract it.

- ✓ **$a \neq 1$:** take a out of the x^2 and x terms first.
- ✓ **Halve:** $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.
- ✓ **Tidy:** combine the constants; multiply back by a if you took it out.
- ✓ **Solve:** isolate the square, square root with $\pm$, then rearrange.
- ✓ **Turning point:** $y = a(x + p)^2 + q$ turns at $(-p, q)$.

Watch out!

The number you subtract is

$\left(\frac{b}{2}\right)^2$, and it is always

subtracted, even when b is negative: $x^2 - 10x = (x - 5)^2 - 25$, not $+25$.

In $2[(x + 3)^2 - 9]$ the 2 multiplies the -9 too, giving -18 . Subtracting only 9 gives the wrong constant -2 .

Square rooting gives two values:

$x - 3 = +\sqrt{5}$ or $-\sqrt{5}$. And the turning point of $(x - 2)^2 - 3$ is at $x = +2$, the opposite sign to the number in the bracket.

11 Linear inequalities and the number line

Syllabus 2.6

Solve an inequality like an equation: do the same thing to every side. The one extra rule: when you multiply or divide by a **NEGATIVE** number, reverse the inequality sign.

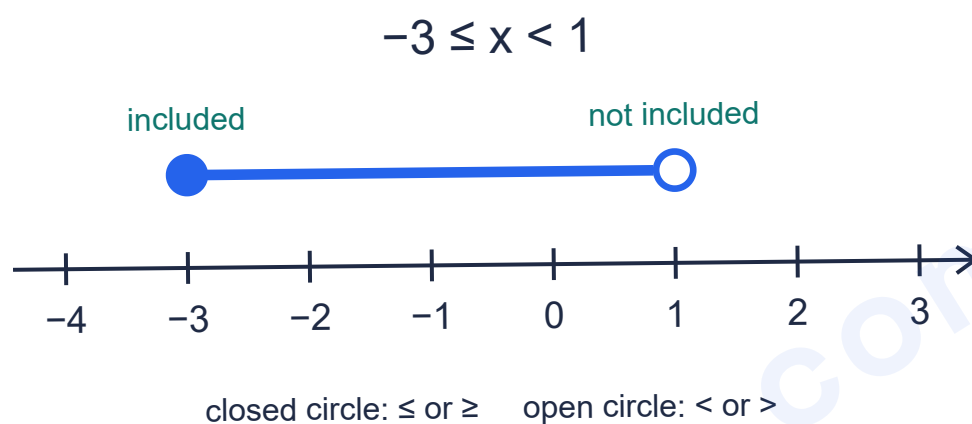


Fig. 1: -3 is included (closed circle) and 1 is not (open circle); every number between them is allowed

Worked example: Draw an inequality

Show $-2 < x \leq 3$ on a number line.

1. Left end: $-2 < x$ is strict, so draw an **OPEN** circle at -2 .
2. Right end: $x \leq 3$ includes 3 , so draw a **CLOSED** circle at 3 .
3. Join the two circles with a line: every number between them satisfies the inequality.

Answer: Open circle at -2 , closed circle at 3 , joined by a line.

*Treat it like an equation,
watch the sign.*

- ✓ **Number line:** open circle for $<$ or $>$; closed circle for $\leq$ or $\geq$.
- ✓ **Solve:** do the same to every side, exactly as for an equation.
- ✓ **Negative:** multiplying or dividing by a negative number reverses the sign.
- ✓ **Double inequality:** work on all three parts at once.
- ✓ **Integers:** check each end point: is it included or not?

Watch out!

Open and closed circles are not decoration. Drawing a closed circle for $x < 3$ says 3 is allowed, and loses the mark. Strict sign, open circle.

Why the flip? $2 < 5$ is true, but multiply both sides by -1 and you get -2 and -5 , where $-2 > -5$. Forgetting to reverse the sign is the most common lost mark on this topic. You can avoid it by moving the x term to the side where it stays positive.

Do not split a double inequality and then only finish one half. Whatever you do to the middle, do to both ends as well, or the answer range is wrong.

In context the answer is often a whole number. Do not round 13.125 to the nearest integer by habit: think about which way keeps the inequality true. For 'at most' round down; for 'at least' round up.

12 Inequalities as regions on a graph

Syllabus 2.6

Draw the boundary as if it were an equation: SOLID for $\leq$ or $\geq$, BROKEN for $<$ or $>$.

Test a point that is not on the line to decide which side is wanted, then shade the side you do NOT want.

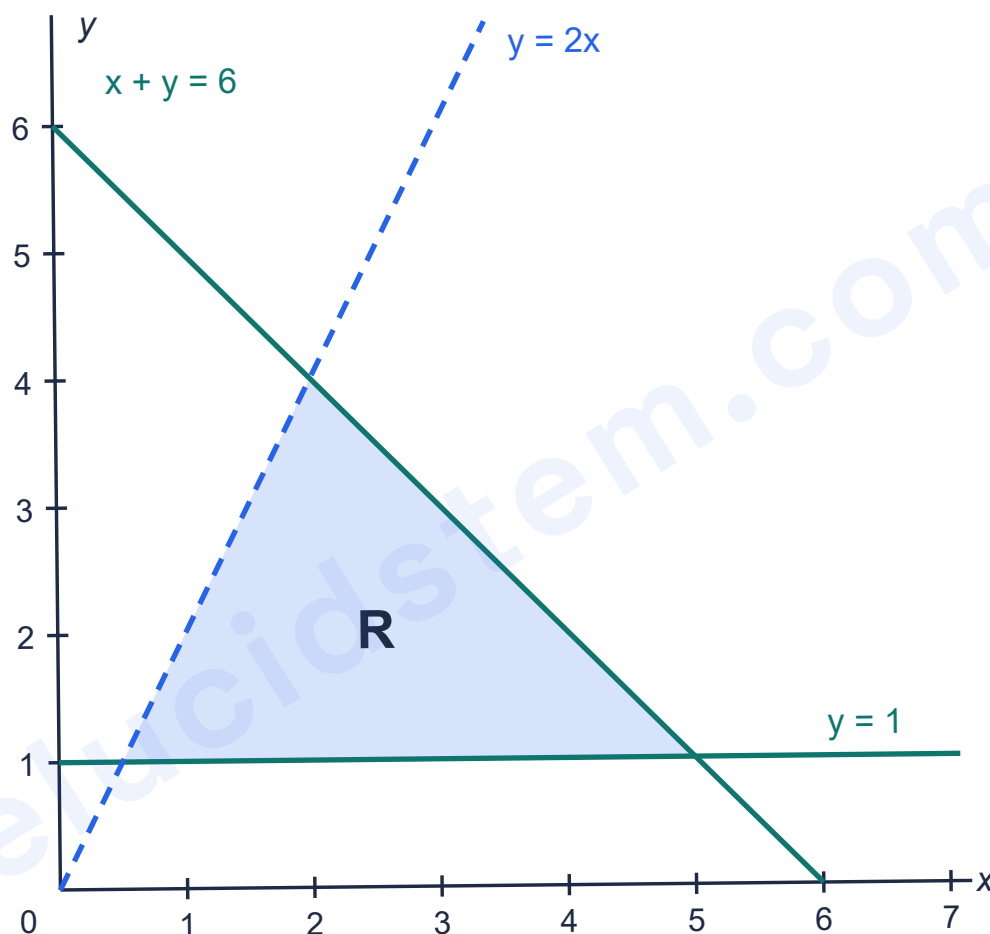


Fig. 1: R satisfies $y \geq 1$, $x + y \leq 6$ and $y < 2x$. The two $\leq / \geq$ boundaries are solid; $y = 2x$ is broken because the inequality is strict. In the exam you shade the three unwanted sides and leave R clear.

Worked example: Vertical and horizontal lines

Show the region $x < 3$ and, on another grid, the region $y \geq -1$.

1. $x < 3$: the boundary $x = 3$ is a VERTICAL line through 3 on the x -axis. Strict sign, so draw it broken.
2. Wanted points have x less than 3, which is the left side. Shade the right side.

3. $y \geq -1$: the boundary $y = -1$ is a HORIZONTAL line. Inclusive sign, so draw it solid.

4. Wanted points are on or above it. Shade below.

Answer: Broken line $x = 3$, shade $x > 3$; solid line $y = -1$, shade $y < -1$.

Line, type, side, shade.

- ✓ **Line:** change the sign to $=$ and draw the line (use intercepts or $y = mx + c$).
- ✓ **Type:** solid for $\leq, \geq$; broken for $<, >$.
- ✓ **Side:** test a point not on the line, usually $(0, 0)$.
- ✓ **Shade:** the unwanted side, leaving the region R clear.
- ✓ **Integer points:** go row by row, and check points on each boundary against its sign.

Watch out!

$x = 3$ is a vertical line, not a horizontal one. Students often draw $x = 3$ across the page because '3 on the x-axis' feels horizontal. A line $x = k$ is always vertical; $y = k$ is always horizontal.

If the line passes through $(0, 0)$, such as $y < 2x$, the origin cannot be a test point: it is ON the line. Use $(1, 0)$ instead: $0 < 2$ is true, so the side containing $(1, 0)$ is wanted.

Points on a broken line are NOT in the region; points on a solid line ARE. Check every boundary point against the sign before listing it.

13 Sequences and the n th term

Syllabus 2.7

If the terms go up by d each time, the n th term is $dn + c$. Find d from the common difference, then c from the first term: $c = \text{first term} - d$.

$n = 1$



4 sticks

$n = 2$



7 sticks

$n = 3$



10 sticks

each new square adds 3 sticks

4, 7, 10, ... n th term = $3n + 1$

Fig. 1: one square needs 4 sticks and each extra square adds 3 (blue), so pattern n uses $3n + 1$ sticks

Worked example: Continue a sequence

Write down the next two terms and the term-to-term rule: 7, 11, 15, 19, ...

1. Differences: $11 - 7 = 4$, $15 - 11 = 4$, $19 - 15 = 4$. Constant, so the rule is add 4.
2. $19 + 4 = 23$, then $23 + 4 = 27$.

Answer: 23, 27; rule: add 4

Difference first, then adjust.

- ✓ **Differences:** subtract each term from the next. Constant means linear.
- ✓ **Start the formula:** the common difference d goes in front of n .
- ✓ **Adjust:** add or subtract so that $n = 1$ gives the first term.
- ✓ **Check:** test $n = 2$ as well.
- ✓ **Use it:** set the formula equal to a number; a whole-number n means it is a term.

Watch out!

A term-to-term rule (add 4) is not an n th term. To find the 100th term with it you would need all 99 terms before. The n th term formula jumps straight there.

For 5, 8, 11, 14 do not write $n + 3$ (adding the difference) or $3n + 5$ (adding the first term). Always check your formula with $n = 1$ AND $n = 2$.

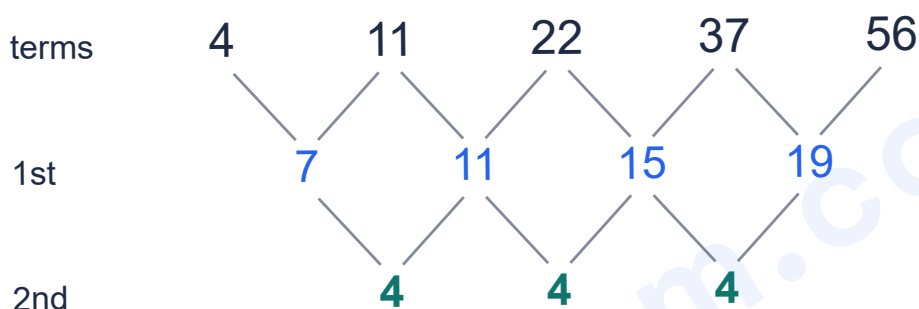
When the answer to n is a decimal, do not round it and claim the number is a term. A decimal n means the number is NOT in the sequence.

A pattern number must be a whole number. Rounding 66.33 is not about 'nearest': round in the direction that keeps you within the 200 sticks.

14 Quadratic, cubic and exponential sequences

Syllabus 2.7

Constant FIRST differences: linear, $dn + c$. Constant SECOND differences: quadratic, and the coefficient of n^2 is half the second difference. A constant RATIO between terms: exponential, $a \times r^{n-1}$.



second difference 4, so the sequence starts $2n^2$

Fig. 1: for 4, 11, 22, 37, 56 the first differences change but the second differences are all 4, so the n th term is $2n^2 + n + 1$

Worked example: Close to the squares

Find the n th term of 2, 5, 10, 17, ...

1. Differences 3, 5, 7 are not constant, so it is not linear.
2. Compare with n^2 : 1, 4, 9, 16. Each term is 1 more.
3. Check $n = 3$: $9 + 1 = 10$. Correct.

Answer: $n^2 + 1$

Differences and ratios tell you the type.

- ✓ **Compare** with n^2 (1, 4, 9, 16) and n^3 (1, 8, 27, 64).
- ✓ **Second differences** constant: quadratic, with a = half the second difference.
- ✓ **Subtract an^2** and find the n th term of what is left.
- ✓ **Constant ratio:** exponential, $a \times r^{n-1}$.
- ✓ **Check** the formula against at least three terms.

Watch out!

n^2 means $n \times n$, not $2n$. For $n = 5$, $n^2 = 25$ but $2n = 10$. Likewise $3n^2$ means $3 \times n^2$, not $(3n)^2$.

The coefficient of n^2 is HALF the second difference. Using the second difference itself ($4n^2$ for Fig. 1) leaves a leftover that is not linear, and the formula fails at $n = 2$.

$3 \times 2^{n-1}$ is NOT 6^{n-1} . The power applies only to the 2; work it out first, then multiply by 3.

Do not cancel or simplify the fractions in the sequence before

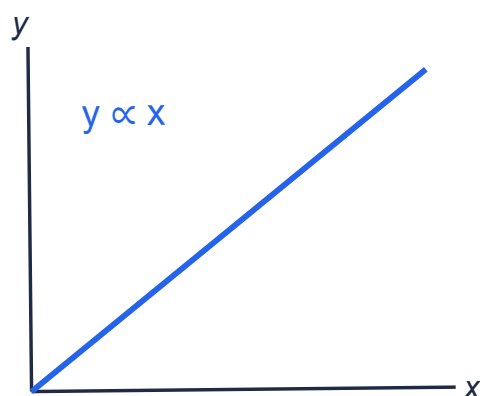
finding the pattern: $\frac{2}{4}$ written as

$\frac{1}{2}$ hides the pattern in the numerators and denominators.

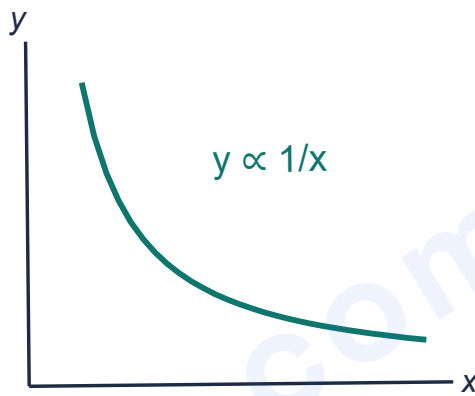
15 Direct and inverse proportion

Syllabus 2.8

Turn $\propto$ into $= k \times$. Direct: $y = kx^n$. Inverse: $y = \frac{k}{x^n}$. Use the given pair of values to find k , write the formula, then use it.



direct: straight line through $(0, 0)$



inverse: xy is constant

Fig. 1: $y = 1.2x$ (direct) is a straight line through the origin; $y = 4/x$ (inverse) falls steeply then levels off, and never meets either axis

Worked example: Linear

y is directly proportional to x . When $x = 4$, $y = 12$. Find y when $x = 7$.

1. Write the equation: $y = kx$.
2. Substitute the known pair: $12 = k \times 4$, so $k = 3$.
3. Formula: $y = 3x$. When $x = 7$, $y = 21$.

Answer: $y = 21$

Equation, constant, formula, answer.

- ✓ **Equation:** replace $\propto$ by $= k \times$ (direct) or $= \frac{k}{\dots}$ (inverse).
- ✓ **Power:** use the exact one in the question: square, cube, square root, cube root.
- ✓ **Constant:** substitute the given pair of values and solve for k .
- ✓ **Formula:** rewrite the equation with the number for k .
- ✓ **Answer:** substitute; if finding x , undo the power at the end.

Watch out!

Always find k and write the formula first. Scaling by eye (' x went from 5 to 3, so...') goes wrong as soon as there is a square or a root involved.

In $y = \frac{k}{\sqrt{x}}$, substitute $\sqrt{9} = 3$, not 9. Read the question for the exact power: 'the square of', 'the square root of' and 'the cube of' all change the equation.

If $y \propto x^2$ and x doubles, y is multiplied by $2^2 = 4$, not 2. The multiplier for y is the multiplier for x raised to the same power.

16 Conversion, distance-time and speed-time graphs

Syllabus 2.9

The gradient of a graph is the rate of change of the vertical quantity per unit of the horizontal one. Distance-time: gradient = speed. Speed-time (Extended): gradient = acceleration, and area under the graph = distance travelled.

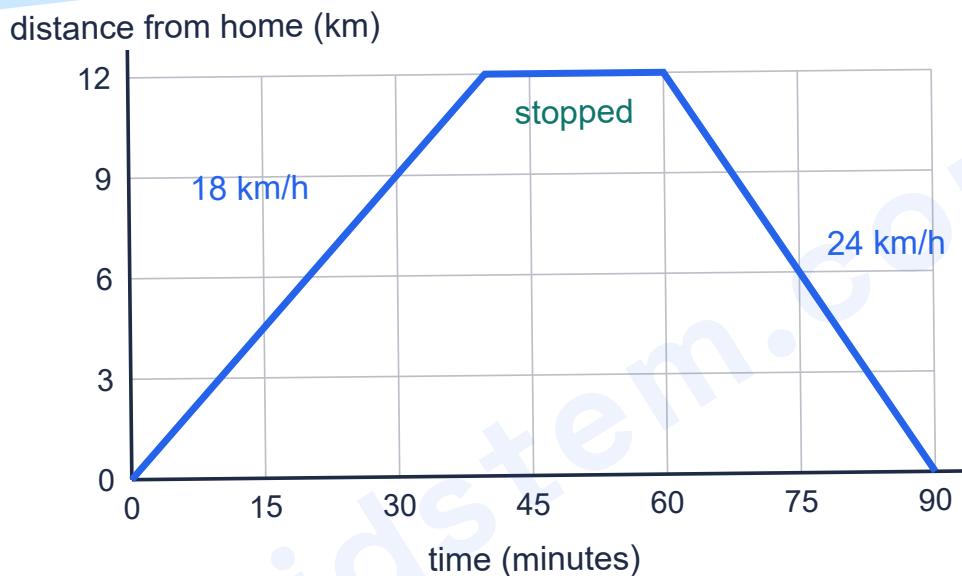


Fig. 1: a cyclist rides 12 km in 40 minutes, rests for 20 minutes, then rides home in 30 minutes

Worked example: Miles and kilometres

A conversion graph is a straight line through (0,0) and (10 miles, 16 km). (a) Convert 25 miles to km. (b) Convert 56 km to miles. (c) What does the gradient represent?

1. Gradient = $\frac{16}{10} = 1.6$ km per mile, so $\text{km} = 1.6 \times \text{miles}$.
2. (a) $25 \times 1.6 = 40$ km.
3. (b) $56 \div 1.6 = 35$ miles.
4. (c) The gradient is the number of kilometres in one mile.

Answer: (a) 40 km (b) 35 miles (c) 1.6 km per mile

Check the axes, then gradient or area.

- ✓ **Axes:** read the quantity, the units and the scale on each axis.
- ✓ **Gradient:** vertical change $\div$ horizontal change, with units (km/h, m/s², \$ per £).
- ✓ **Units:** change minutes to hours before working out km/h.
- ✓ **Area (Extended):** split the area under a speed-time graph into triangles, rectangles and trapezia.
- ✓ **Curves (Extended):** draw a tangent and use two far-apart points on it.

Watch out!

Read the scale on each axis before you read off a value. On a graph where one small square is 2 units, a point three squares along is 6, not 3.

Time read in minutes must be changed to hours for km/h: 40 minutes is $\frac{2}{3}$ h, not 0.4 h. And the average speed is NOT the mean of the stage speeds: $\frac{18 + 24}{2} = 21$ is wrong because the stages last different times and the stop counts.

Check which graph you have. On a distance-time graph the gradient is speed; on a speed-time graph the gradient is acceleration and a flat line means steady speed. Using the wrong meaning is the commonest lost mark here.

Do not use two points on the curve: that gives the AVERAGE speed between them, not the speed at one instant. Here the average speed from 0 to 4 s is $16 \div 4 = 4$ m/s, only half the speed at $t = 4$.

17 Graphs of functions: tables, plotting and tangents

Syllabus 2.10

Substitute each x into the equation (brackets round negatives), plot every point, then join them with ONE smooth curve. The gradient of a curve at a point is the gradient of the tangent drawn there.

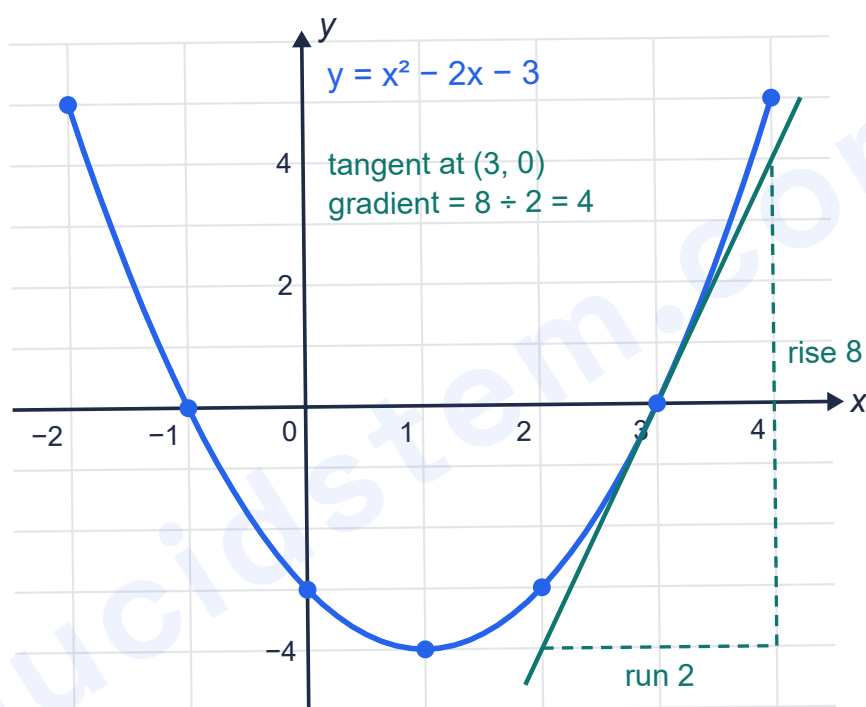


Fig. 1: the table of values for $y = x^2 - 2x - 3$ plotted from $x = -2$ to 4, with the tangent at $(3, 0)$; rise 8 over run 2 gives a gradient of 4

Worked example: Completing a table for a quadratic

Complete the table for $y = x^2 - 2x - 3$ at $x = -2$ and $x = 1$.

1. $x = -2$: $(-2)^2 - 2(-2) - 3 = 4 + 4 - 3$. The square of a negative is positive, and $-2 \times -2 = +4$.
2. So $y = 5$.
3. $x = 1$: $1^2 - 2(1) - 3 = 1 - 2 - 3 = -4$.
4. Check with symmetry: the values 5, 0, -3, -4, -3, 0, 5 are symmetrical about $x = 1$, as a quadratic should be.

Answer: $y = 5$ at $x = -2$; $y = -4$ at $x = 1$

Table, plot, smooth curve, then read.

- ✓ **Substitute carefully:** put negative x values in brackets, especially when squaring or cubing.
- ✓ **Check the pattern:** quadratic tables are symmetrical; a value that breaks the pattern is usually a slip.
- ✓ **Plot and join:** one smooth curve through every point; never join reciprocal branches across $x = 0$.
- ✓ **Tangent:** rule a line touching the curve at the point, then $\text{gradient} = \text{rise} \div \text{run}$ using a big triangle.
- ✓ **Exponential (Extended):** ab^x starts at a when $x = 0$ and is multiplied by b for each step of 1.

Watch out!

Typing -2^2 into a calculator gives -4 , because it squares 2 and then makes it negative. Always type $(-2)^2$ with brackets: the answer is $+4$. One wrong value ruins the shape of the whole curve.

Joining the points of a curve with straight ruler lines loses the mark for the curve. Use one smooth freehand curve that passes through every point, and do not flatten the bottom of a parabola between two equal points.

Exponential decay is NOT a straight line: after 3 years the car is worth \$6144, not $12\,000 - 3 \times 2400 = \$4800$. The 20% is taken from a smaller value each year.

Use two points on the TANGENT that are far apart, not points on the curve, and read each scale carefully: if the y -axis goes up in 2s, a rise of 4 squares is 8, not 4.

18 Solving equations graphically

Syllabus 2.10

The solutions of curve = line are the x -coordinates of the points where the two graphs cross. To solve a new equation with a curve you already have, rearrange it so one side is exactly the curve's equation; the other side is the line to draw.

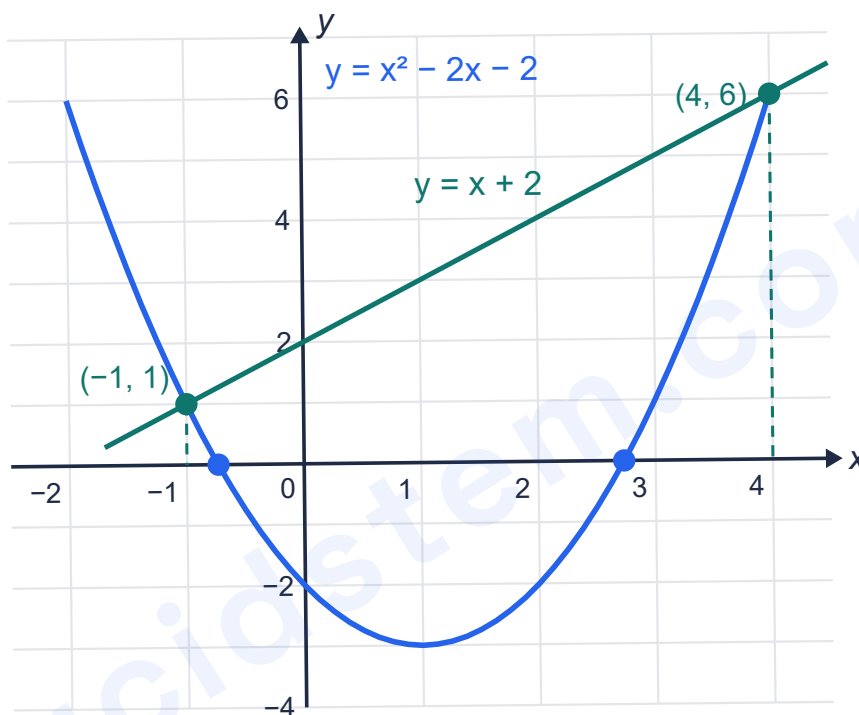


Fig. 1: $y = x^2 - 2x - 2$ crosses the x -axis at about -0.7 and 2.7 , and meets the line $y = x + 2$ at $(-1, 1)$ and $(4, 6)$

Worked example: Roots from the graph

Use Fig. 1 to solve $x^2 - 2x - 2 = 0$.

1. The left side is exactly the curve $y = x^2 - 2x - 2$, and the right side is 0, so the line to use is $y = 0$; the x -axis.
2. The curve crosses the x -axis twice.
3. Read down at each crossing: $x \approx -0.7$ and $x \approx 2.7$.

Answer: $x \approx -0.7$ or $x \approx 2.7$ (exact values $1 \pm \sqrt{3}$)

Match the curve, draw the line, read the x -values.

- ✓ **Roots:** $f(x) = 0$ is solved where $y = f(x)$ meets the x -axis.
- ✓ **Equal to a number:** $f(x) = k$ is solved where the curve meets the horizontal line $y = k$.
- ✓ **New equation:** add or subtract the same terms on both sides until one side is exactly the drawn curve.
- ✓ **The other side** is the line to draw; plot it with two or three points.
- ✓ **Answer** with the x -coordinates of every crossing, to the accuracy of the graph (usually 1 d.p.).

Watch out!

Give the x -values, not the y -value. For $x^2 - 2x - 2 = 3$, the answer is NOT $y = 3$; it is the two x -coordinates where the line $y = 3$ cuts the curve. And give BOTH solutions.

When you rearrange, change BOTH sides by the same amount.

Students often move terms across and write $y = 3x + 1$ for this question, but the line must come from keeping the equation balanced: $y = x - 1$.

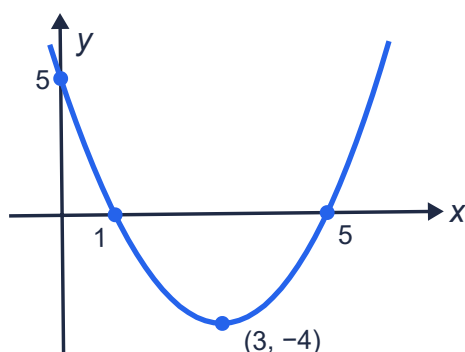
Only read solutions inside the drawn range. If a line leaves the grid before meeting the curve again, say there are no more solutions in that range rather than guessing.

19 Sketching curves: roots, intercepts and turning points

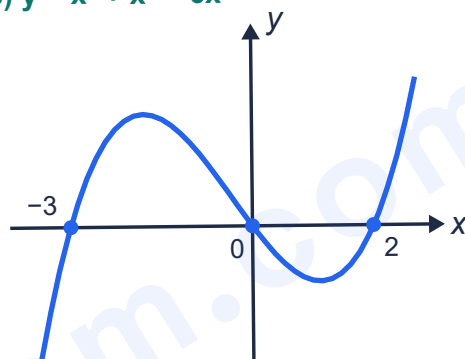
Syllabus 2.11

Sketch = correct shape + labelled key points. Set $x = 0$ for the y -intercept and $y = 0$ for the roots; for a quadratic the line of symmetry is halfway between the roots, and completing the square gives the turning point.

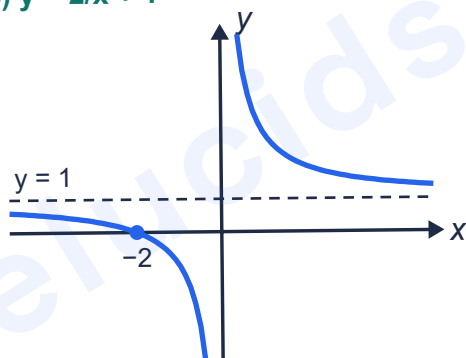
(a) $y = x^2 - 6x + 5$



(b) $y = x^3 + x^2 - 6x$



(c) $y = 2/x + 1$



(d) $y = 2^x - 2$

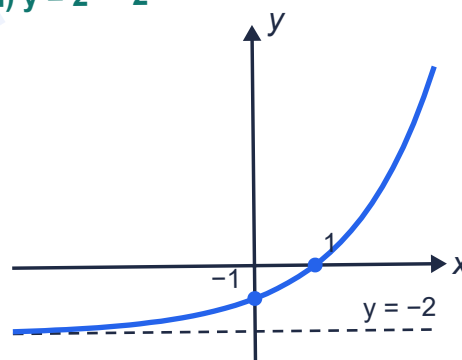


Fig. 1: four sketches with their key points: (a) roots 1 and 5, minimum (3, -4); (b) roots -3, 0 and 2; (c) asymptotes $x = 0$ and $y = 1$; (d) asymptote $y = -2$, through (0, -1) and (1, 0)

Worked example: Sketching a straight line

Sketch $2x + 3y = 12$.

1. $x = 0$: $3y = 12$, so $y = 4$. The line meets the y -axis at (0, 4).
2. $y = 0$: $2x = 12$, so $x = 6$. It meets the x -axis at (6, 0).
3. Join the two points with a ruled line: it slopes down from left to right.

Answer: A straight line through (0, 4) and (6, 0)

Shape, intercepts, turning points, asymptotes.

- ✓ **Shape:** decide from the highest power and its sign (U or $\cap$; rising or falling cubic; two-branch reciprocal; exponential).
- ✓ **y-intercept:** put $x = 0$.
- ✓ **Roots:** put $y = 0$ and solve, usually by factorising.
- ✓ **Quadratic turning point (Extended):** complete the square, $y = (x + p)^2 + q$ turns at $(-p, q)$.
- ✓ **Asymptotes (Extended):** dashed lines, labelled; the curve gets close but never touches.
- ✓ **Label** every key point with its coordinates.

Watch out!

In $(x - 3)^2 - 4$ the turning point is $(3, -4)$, not $(-3, -4)$. The sign inside the bracket flips; the number outside does not.

Do not forget the root $x = 0$ when you take out a factor of x . $x^3 + x^2 - 6x = 0$ has THREE solutions, not two.

$y = 2^x$ does NOT pass through the origin: $2^0 = 1$. And an exponential curve never crosses its asymptote, so your sketch must not touch the dashed line.

20 Differentiation: gradient functions

Syllabus 2.12

If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$; multiply by the power, then take one off the power. A constant term differentiates to 0. Substitute an x -value into $\frac{dy}{dx}$ to get the gradient there.

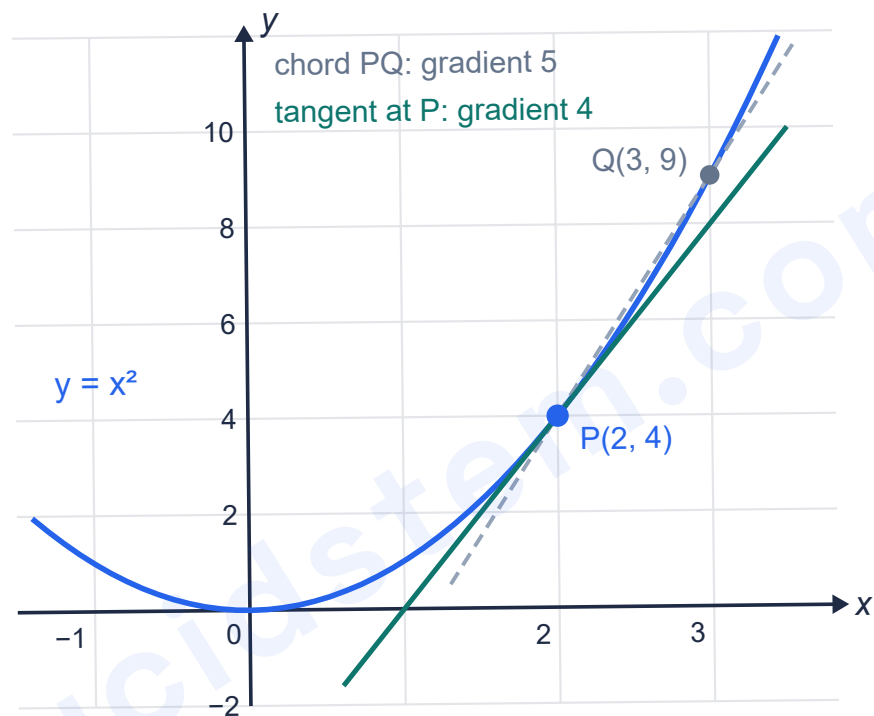


Fig. 1: on $y = x^2$, the chord from $P(2, 4)$ to $Q(3, 9)$ has gradient 5; as Q slides towards P the chord turns into the tangent at P , gradient 4

Worked example: Spotting the gradient function

Using chords and tangents on $y = x^2$, the gradient at $x = 2$ is 4. The same method gives gradient 2 at $x = 1$ and 6 at $x = 3$. Find a rule.

1. List the results: $x = 1 \rightarrow 2$, $x = 2 \rightarrow 4$, $x = 3 \rightarrow 6$.
2. Each gradient is double the x -value.
3. So the gradient of $y = x^2$ at any point is $2x$.

Answer: Gradient of $y = x^2$ is $2x$: this is its gradient function, $\frac{dy}{dx} = 2x$.

Expand, differentiate each term, substitute.

- ✓ **Prepare:** expand any brackets so y is a sum of terms ax^n .
- ✓ **Rule:** $ax^n \rightarrow anx^{n-1}$; $ax \rightarrow a$; a constant $\rightarrow 0$.
- ✓ **Notation:** write $\frac{dy}{dx} = \dots$ for the gradient function.
- ✓ **Gradient at a point:** substitute the x -coordinate into $\frac{dy}{dx}$.
- ✓ **Point with a given gradient:** solve $\frac{dy}{dx} = k$, then find y from the original equation.

Watch out!

A tangent touches the curve at one point and has the same steepness there. A chord joins two points of the curve and only approximates the gradient: do not stop at the chord.

Do not differentiate each bracket and multiply: for $(x - 4)(2x + 1)$ that gives $1 \times 2 = 2$, which is wrong. Expand first, then differentiate term by term.

Substitute into $\frac{dy}{dx}$ for the gradient, but into y for the coordinate. Putting $x = 2$ into $y = x^3 - 2x + 1$ gives 5, the height of the point, not its gradient.

21 Stationary points

Syllabus 2.12

Stationary points are where $\frac{dy}{dx} = 0$. Solve for x , then find y from the original equation. If $\frac{d^2y}{dx^2} > 0$ it is a minimum; if $\frac{d^2y}{dx^2} < 0$ it is a maximum.

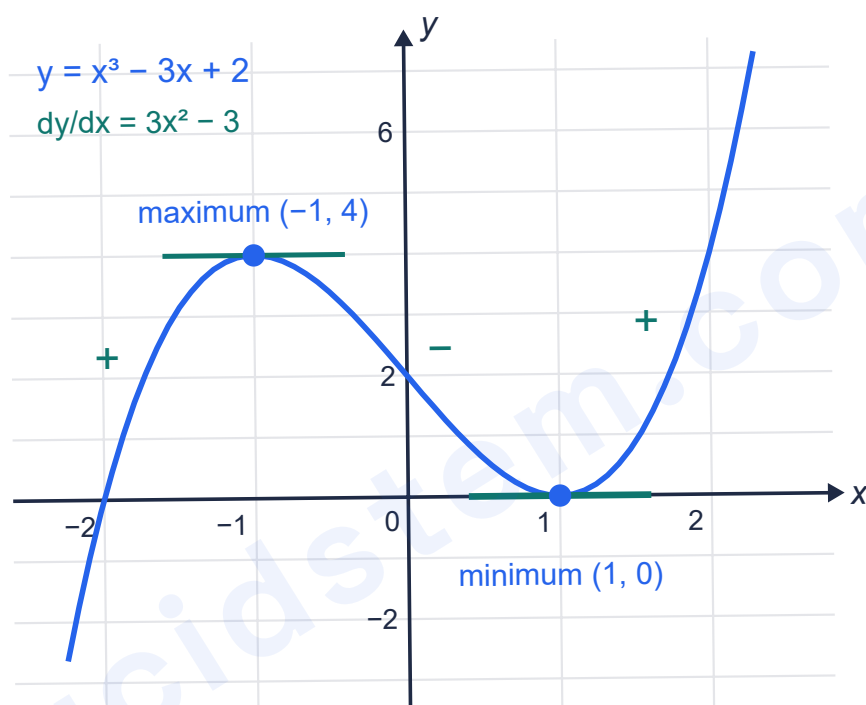


Fig. 1: $y = x^3 - 3x + 2$ has horizontal tangents at the maximum $(-1, 4)$ and the minimum $(1, 0)$; the gradient goes + then - at the maximum and - then + at the minimum

Worked example: A quadratic

Find the stationary point of $y = x^2 - 8x + 3$.

1. $\frac{dy}{dx} = 2x - 8$.

2. $2x - 8 = 0$, so $x = 4$.

3. $y = 4^2 - 8(4) + 3 = 16 - 32 + 3 = -13$.

Answer: $(4, -13)$. The curve is U-shaped, so this is a minimum.

Differentiate, set to zero, find y , classify.

- ✓ **Differentiate:** expand first if needed, then find $\frac{dy}{dx}$.
- ✓ **Solve** $\frac{dy}{dx} = 0$; a cubic gives a quadratic, so look for two answers.
- ✓ **Coordinates:** substitute each x into the original equation.
- ✓ **Classify:** $\frac{d^2y}{dx^2} > 0$ minimum, < 0 maximum; or check the gradient either side.
- ✓ **In context:** build a one-variable formula, reject impossible values, and answer the actual question.

Watch out!

$x^2 = 1$ has TWO solutions, $x = 1$ and $x = -1$. Missing the negative root loses a whole stationary point.

Decide using the second derivative or the gradient, not the size of y . Here the maximum $(1, 5)$ is only slightly higher than the minimum $(2, 4)$, and on other curves a local maximum can even be LOWER than a minimum.

Answer the question that was asked. Finding $x = 10$ is not the end: the question wants the AREA, 200 m^2 . Also reject values that make no physical sense, such as $x = 6$ for the box.

22 Functions: notation, inverse and composite

Syllabus 2.13

$f(a)$ means substitute a into the rule. f^{-1} undoes f : write $y = f(x)$ and rearrange to make x the subject. $gf(x) = g(f(x))$: apply f FIRST, then g .

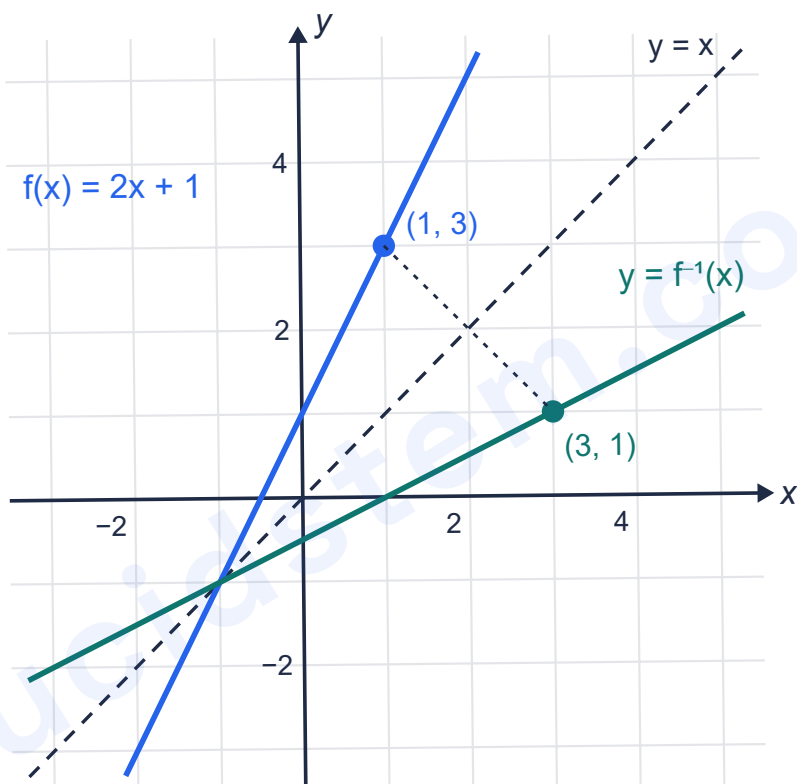


Fig. 1: $f(x) = 2x + 1$ and its inverse $f^{-1}(x) = (x - 1)/2$ are reflections of each other in the line $y = x$; $f(1) = 3$, so $f^{-1}(3) = 1$

Worked example: Evaluating and solving

$f(x) = 3x - 5$. Find (a) $f(-2)$ (b) the value of x for which $f(x) = 10$.

- (a) $f(-2) = 3(-2) - 5 = -6 - 5 = -11$.
- (b) Set the rule equal to 10: $3x - 5 = 10$.
- $3x = 15$, so $x = 5$.

Answer: (a) -11 (b) $x = 5$

Substitute, rearrange, apply in order.

- ✓ **$f(a)$:** replace every x in the rule with a , using brackets.
- ✓ **Solve $f(x) = k$:** set the rule equal to k and solve.
- ✓ **Inverse:** write $y = f(x)$, make x the subject, then write $f^{-1}(x)$.
- ✓ **Composite:** $gf(x) = g(f(x))$, so the function nearest x acts first.
- ✓ **Simplify** algebraic composites fully, expanding brackets in denominators.

Watch out!

In $2x^2$ only the x is squared:

$$h(-2) = 2 \times (-2)^2 + 3 = 2 \times 4 + 3 = 11, \text{ not } (2 \times -2)^2 + 3 = 19.$$

$f^{-1}(x)$ is NOT $\frac{1}{f(x)}$. The -1 means 'inverse', not a power: for $f(x) = 3x - 5$, $f^{-1}(x) = \frac{x+5}{3}$, not $\frac{1}{3x-5}$.

$gf(x)$ means f first, even though g is written first. It is also not a product: $gf(x) \neq g(x) \times f(x)$.