

Trigonometry

IGCSE Maths 0580 · Topic 6 revision notes

Everything from the 10 lessons of this topic on a few pages: the key idea of each lesson, the diagram to remember, the points to learn and the traps examiners set. Read it, cover it, test yourself.

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| 1. <u>Pythagoras' theorem</u> | 6. <u>Solving trigonometric equations</u> |
| 2. <u>Right-angled trigonometry: SOH CAH TOA</u> | 7. <u>The sine rule</u> |
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| 4. <u>Exact trigonometric values</u> | 9. <u>Area of a triangle: half $ab \sin C$</u> |
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1 Pythagoras' theorem

Syllabus 6.1

In a right-angled triangle with hypotenuse c (the side opposite the right angle) and shorter sides a and b : $a^2 + b^2 = c^2$. To find the hypotenuse, add the squares; to find a shorter side, subtract.

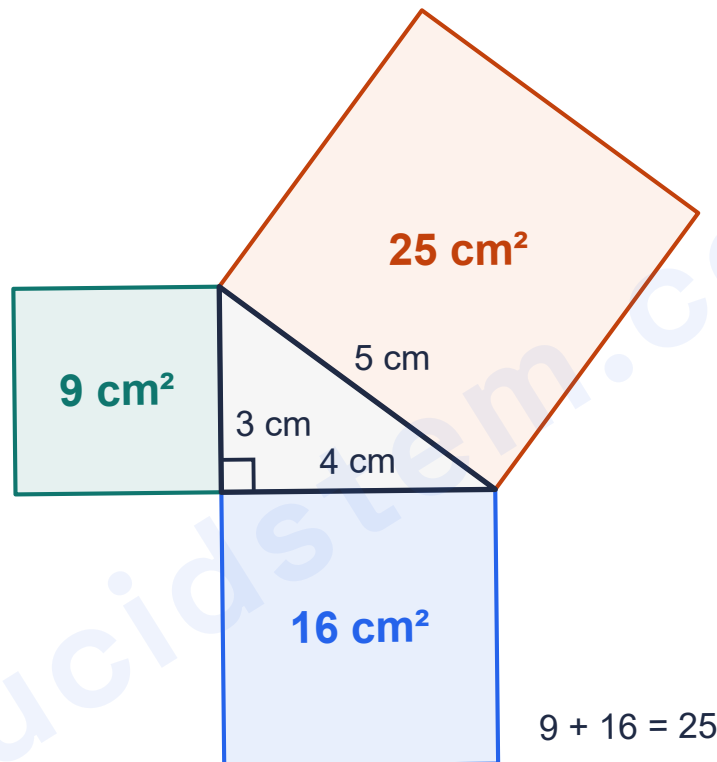


Fig. 1: a 3-4-5 triangle drawn to scale. The squares on the two shorter sides (9 cm^2 and 16 cm^2) together equal the square on the hypotenuse (25 cm^2)

Worked example: Finding the hypotenuse

A right-angled triangle has shorter sides 7 cm and 11 cm. Find the length of the hypotenuse.

1. The hypotenuse is the unknown, so add the squares: $c^2 = 7^2 + 11^2$.
2. $c^2 = 49 + 121 = 170$.
3. Square root: $c = \sqrt{170} = 13.038 \dots$

Answer: 13.0 cm (3 s.f.). Check: longer than 11 cm, as it must be.

*Find the hypotenuse first,
then decide: add or subtract.*

- ✓ **Sketch:** draw the right-angled triangle and mark the right angle.
- ✓ **Label:** the hypotenuse c is opposite the right angle.
- ✓ **Hypotenuse missing:** $c^2 = a^2 + b^2$, then square root.
- ✓ **Shorter side missing:** $a^2 = c^2 - b^2$, then square root.
- ✓ **Check:** the hypotenuse must be the longest side; round to 3 s.f. at the end.

Watch out!

Adding the squares when you need a shorter side. If the hypotenuse is given, you must SUBTRACT.

Using $6.5^2 + 1.8^2$ for the ladder gives 6.74 m, longer than the ladder itself, which is impossible.

For the isosceles triangle, use HALF the base. Using the full 10 cm gives $\sqrt{169 - 100} = 8.31$ cm, which is wrong.

Rounding too early. If a first length is $\sqrt{97} = 9.848\dots$, keep it in the calculator for the next step; using 9.8 can change the final 3 s.f. answer.

2 Right-angled trigonometry: SOH CAH TOA

Syllabus 6.2

Label the sides from the angle: hypotenuse (opposite the right angle), opposite (facing the angle), adjacent (next to the angle). Then $\sin \theta = \frac{\text{opp}}{\text{hyp}}$, $\cos \theta = \frac{\text{adj}}{\text{hyp}}$, $\tan \theta = \frac{\text{opp}}{\text{adj}}$.

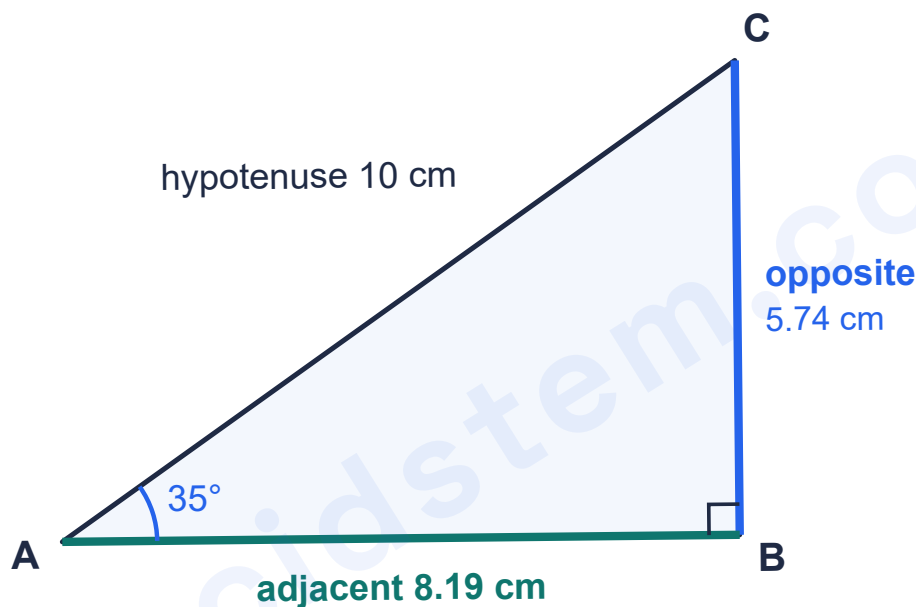


Fig. 1: sides labelled from the 35° angle at A, drawn to scale. $BC = 10 \sin 35^\circ = 5.74 \text{ cm}$ and $AB = 10 \cos 35^\circ = 8.19 \text{ cm}$

Worked example: Unknown on top

In Fig. 1, $AC = 10 \text{ cm}$ and angle $A = 35^\circ$. Find BC .

1. From angle A: AC is the hypotenuse and BC is opposite. Opposite and hypotenuse means SOH.
2. $\sin 35^\circ = \frac{BC}{10}$.
3. Multiply both sides by 10: $BC = 10 \sin 35^\circ = 5.7357 \dots$

Answer: $BC = 5.74 \text{ cm}$ (3 s.f.)

Label from the angle, then pick the ratio.

- ✓ **Label:** hypotenuse first (opposite the right angle), then opposite and adjacent from the angle.
- ✓ **Choose:** SOH, CAH or TOA, whichever uses the two sides in the question.
- ✓ **Unknown on top:** multiply. **Unknown on the bottom:** divide.
- ✓ **Angle:** work out the ratio, then use $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$.
- ✓ **Accuracy:** sides to 3 s.f., angles to 1 d.p.; keep full values between steps.

Watch out!

Writing $h = 6 \cos 52^\circ = 3.69$ cm. When the unknown is on the bottom of the fraction you divide, not multiply. A 'hypotenuse' shorter than the other side is a sure sign of this slip.

Using the full base (10 cm) in the isosceles triangle gives $\cos^{-1}(10/12) = 33.6^\circ$. The right angle is only in the half-triangle, so use half the base.

Rounding AD too early. With $AD = 5.8$ you only just survive ($DC = 12.44$); with $AD = 6$ you get $DC = 12.9$ cm and lose the accuracy mark. Keep the unrounded value in your calculator.

3 Elevation, depression and bearings

Syllabus 6.2

Angles of elevation and depression are measured from the HORIZONTAL, and by alternate angles they are equal. Bearings are measured clockwise from NORTH and written with three figures. Find the right-angled triangle, then use SOH CAH TOA or Pythagoras.

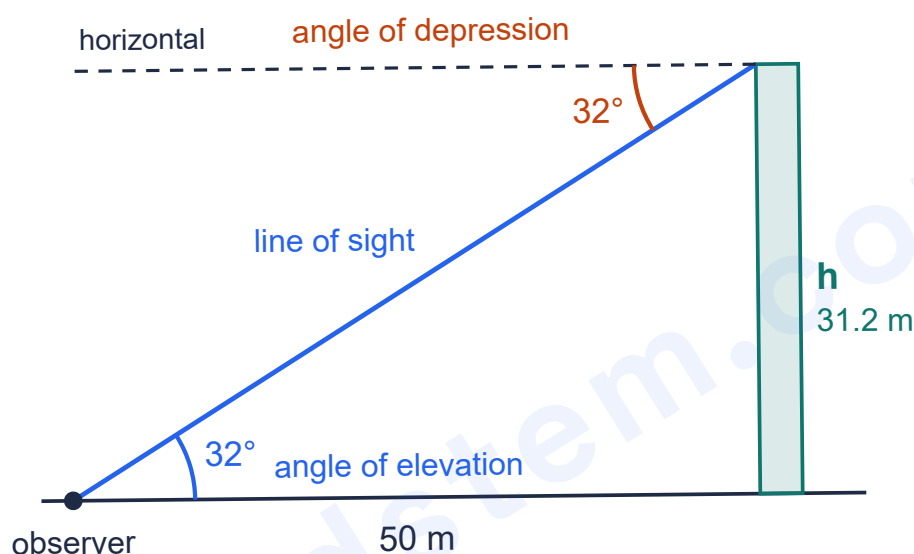


Fig. 1: drawn to scale. The angle of elevation from the observer and the angle of depression from the top are both 32° (alternate angles), and $h = 50 \tan 32^\circ = 31.2 \text{ m}$

Worked example: Height from an angle of elevation (Extended)

From a point on level ground 50 m from the foot of a tower, the angle of elevation of the top is 32° . Find the height of the tower.

1. The height is opposite the 32° angle and the 50 m is adjacent: TOA.

2. $\tan 32^\circ = \frac{h}{50}$, so $h = 50 \tan 32^\circ = 31.243 \dots$

Answer: 31.2 m (3 s.f.)

Sketch it, find the right angle, then trigonometry.

- ✓ **Sketch:** draw a clear diagram with the horizontal ground or north lines marked.
- ✓ **Elevation and depression:** both measured from the horizontal, and equal by alternate angles.
- ✓ **Bearings:** clockwise from north, three figures; back bearing is $\pm 180^\circ$.
- ✓ **Journeys:** split into north-south and east-west parts, or spot a right-angle turn.
- ✓ **Shortest distance:** the perpendicular from the point to the line.

Watch out!

Measuring the angle of depression from the vertical cliff face. It is always measured from the horizontal. If you put 18° at the top of the cliff against the vertical, you get $80 \tan 18^\circ = 26.0$ m, which is far too close.

Measuring anticlockwise, or from the wrong point. 'The bearing of B from A ' means stand at A , face north, turn clockwise until you face B .

Using the distance along the path, $12 \cos 20^\circ = 11.3$ km, or the original 12 km. The shortest distance is the perpendicular one, and it is always less than the distance to H .

4 Exact trigonometric values

Syllabus 6.3

Half a square gives sides 1, 1, $\sqrt{2}$ and the 45° values. Half an equilateral triangle of side 2 gives sides 1, $\sqrt{3}$, 2 and the 30° and 60° values. Read off SOH CAH TOA from these triangles.

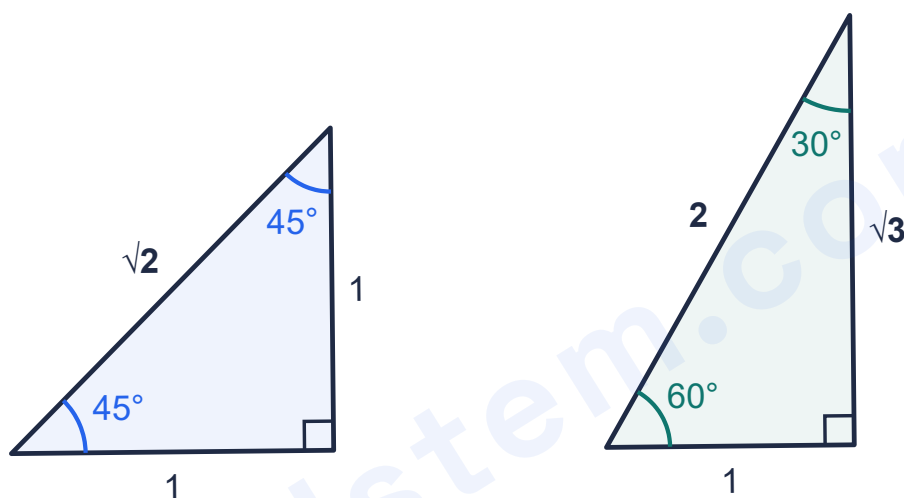


Fig. 1: the two special triangles, drawn to scale. Left: half a unit square (1, 1, $\sqrt{2}$). Right: half an equilateral triangle of side 2 (1, $\sqrt{3}$, 2)

Worked example: Reading values from a triangle

Use the triangles in Fig. 1 to find the exact values of $\sin 60^\circ$, $\cos 60^\circ$ and $\tan 60^\circ$.

1. In the right-hand triangle, from the 60° angle: opposite = $\sqrt{3}$, adjacent = 1, hypotenuse = 2.

$$2. \sin 60^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}, \cos 60^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}.$$

$$3. \tan 60^\circ = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

Answer: $\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$

Sketch the special triangle, then read off the ratio.

- ✓ **45°:** half a square, sides 1, 1, $\sqrt{2}$.
- ✓ **30° and 60°:** half an equilateral triangle, sides 1, $\sqrt{3}$, 2.
- ✓ **0° and 90°:** sin goes $0 \rightarrow 1$, cos goes $1 \rightarrow 0$.
- ✓ **Squares:** $\sin^2 x = (\sin x)^2$; square top and bottom.
- ✓ **Surds:** $\sqrt{3} \times \sqrt{3} = 3$; rationalise if asked.

Watch out!

Mixing up $\sin 30^\circ$ and $\sin 60^\circ$.
Sine grows with the angle, so the smaller angle has the smaller

value: $\sin 30^\circ = \frac{1}{2} <$

$\sin 60^\circ = \frac{\sqrt{3}}{2}$.

$\sin^2 60^\circ$ means $(\sin 60^\circ)^2 =$

$\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$. It is not

$\sin(60^2)$ and not $\frac{\sqrt{3}}{4}$: square the top AND the bottom.

Dividing by a fraction: $\frac{6}{\frac{1}{2}} = 12$,

not 3. Dividing by $\frac{1}{2}$ is the same as multiplying by 2.

5 Graphs of sine, cosine and tangent

Syllabus 6.4

$y = \sin x$ and $y = \cos x$ are waves between -1 and 1 that repeat every 360° . $y = \tan x$ repeats every 180° and has vertical asymptotes at 90° and 270° . Their symmetry gives a second angle with the same value: $\sin(180^\circ - x) = \sin x$, $\cos(360^\circ - x) = \cos x$, $\tan(180^\circ + x) = \tan x$.

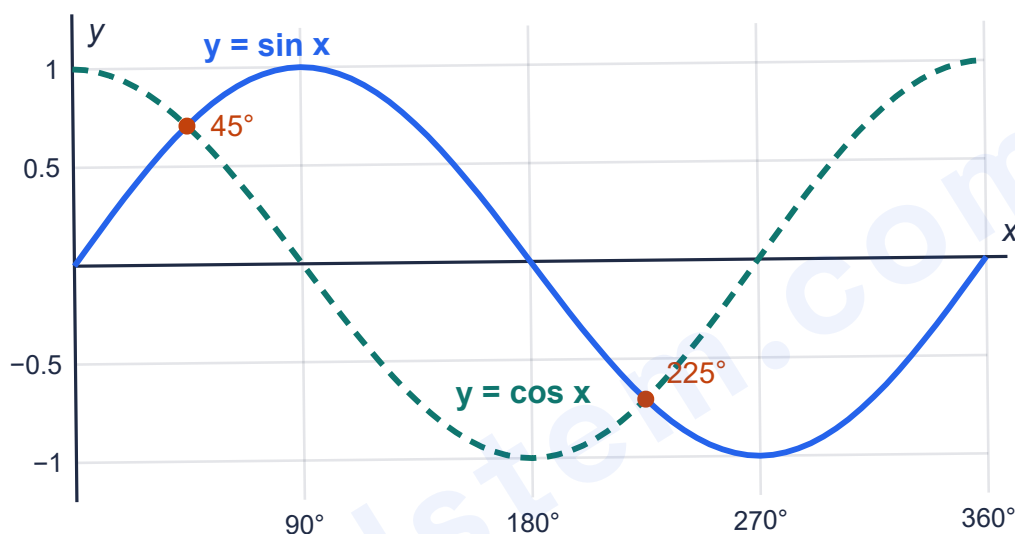


Fig. 1: $y = \sin x$ (solid) and $y = \cos x$ (dashed) for 0° to 360° , plotted from computed values. The graphs cross where $\sin x = \cos x$, at 45° and 225°

Worked example: A table of values

Complete a table of values of $y = \cos x$ for $x = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ, 360^\circ$, and describe the shape.

1. $\cos 0^\circ = 1$, $\cos 60^\circ = 0.5$, $\cos 120^\circ = -0.5$, $\cos 180^\circ = -1$.
2. $\cos 240^\circ = -0.5$, $\cos 300^\circ = 0.5$, $\cos 360^\circ = 1$.
3. The values fall from 1 to -1 and rise back to 1, symmetrical about $x = 180^\circ$.

Answer: 1, 0.5, -0.5 , -1 , -0.5 , 0.5, 1: a smooth U-shaped wave (a curve, never straight lines between the points).

Sketch the graph first, then read across.

- ✓ **Sine:** starts at 0, peaks at $(90^\circ, 1)$, lowest at $(270^\circ, -1)$.
- ✓ **Cosine:** starts at 1, lowest at $(180^\circ, -1)$, back to 1 at 360° .
- ✓ **Tangent:** zeros at $0^\circ, 180^\circ, 360^\circ$; asymptotes at $90^\circ, 270^\circ$; period 180° .
- ✓ **Second angle:** $180^\circ - \alpha$ (sin), $360^\circ - \alpha$ (cos), $180^\circ + \alpha$ (tan).
- ✓ **Count solutions:** draw the horizontal line and count the crossings.

Watch out!

Sketching sine and cosine with sharp corners at the top and bottom, or as straight zig-zags. The graphs are smooth curves that turn gently at their maximum and minimum.

Drawing the tangent curve across the asymptote at 90° . Each branch climbs up towards $+\infty$ on the left of the asymptote and comes back from $-\infty$ on the right.

Using the sine pattern $180^\circ - \alpha$ for cosine. For cosine the partner of 70° is 290° ($360^\circ - 70^\circ$), not 110° . Always check with a quick sketch.

6 Solving trigonometric equations

Syllabus 6.4

Rearrange to $\sin x = k$, $\cos x = k$ or $\tan x = k$. Use the inverse key for the first angle, then the graph's symmetry for the rest: $\sin$: $180^\circ - \alpha$; $\cos$: $360^\circ - \alpha$; $\tan$: $\alpha + 180^\circ$. Keep only answers in $0^\circ \leq x \leq 360^\circ$.

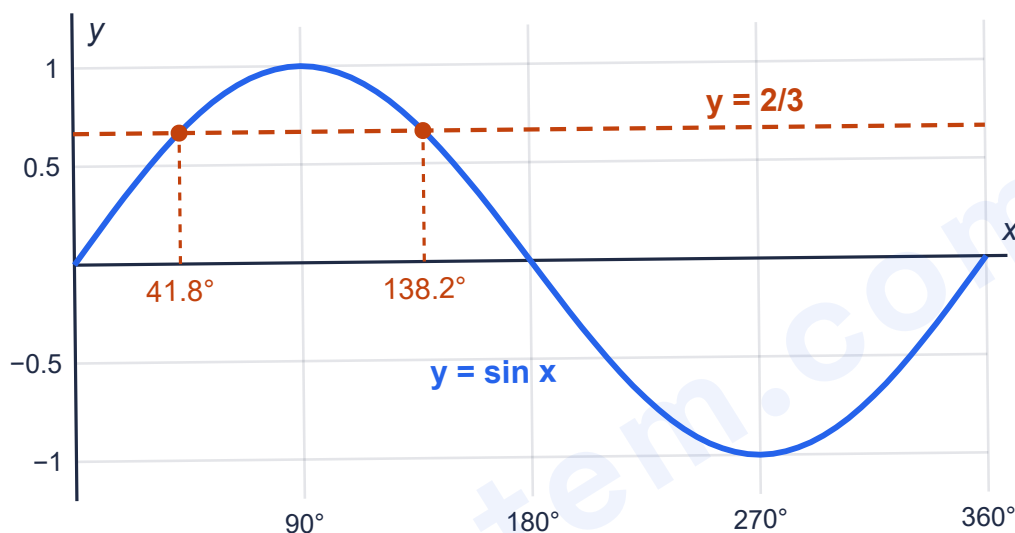


Fig. 1: the line $y = 2/3$ crosses $y = \sin x$ twice between 0° and 360° , at $x = 41.8^\circ$ and $x = 180^\circ - 41.8^\circ = 138.2^\circ$ (plotted from computed values)

Worked example: Solving a sine equation

Solve $\sin x = \frac{2}{3}$ for $0^\circ \leq x \leq 360^\circ$.

1. First angle: $x = \sin^{-1}\left(\frac{2}{3}\right) = 41.81\dots^\circ$.
2. The sine graph is symmetrical about $x = 90^\circ$, so the second solution is $180^\circ - 41.81^\circ = 138.19^\circ$.
3. Sine is negative from 180° to 360° , so there are no more solutions.

Answer: $x = 41.8^\circ$ or $x = 138.2^\circ$

Rearrange, inverse, then use the graph.

- ✓ **Rearrange** to $\sin x = k$, $\cos x = k$ or $\tan x = k$.
- ✓ **Check** that $-1 \leq k \leq 1$ for sine and cosine; otherwise no solutions.
- ✓ **First angle** from $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$.
- ✓ **Other angles:** $180^\circ - \alpha$ (sin), $360^\circ - \alpha$ (cos), $\alpha + 180^\circ$ (tan).
- ✓ **Range:** keep only $0^\circ \leq x \leq 360^\circ$; give answers to 1 d.p.

Watch out!

Stopping after the calculator answer. 'Solve for $0^\circ \leq x \leq 360^\circ$ ' almost always has two answers, and the second one is a mark on its own.

Giving -63.4° as an answer. It solves the equation but is outside $0^\circ \leq x \leq 360^\circ$, so it earns nothing. Use it only as a stepping stone.

Using the pattern for the wrong function. For $\cos x = 0.25$, the partner of 75.5° is $360^\circ - 75.5^\circ$, not $180^\circ - 75.5^\circ$. A quick sketch of the right graph prevents this.

7 The sine rule

Syllabus 6.5

In any triangle ABC , with side a opposite angle A and so on: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. Use it when you know a side AND its opposite angle. To find an angle, flip it: $\frac{\sin A}{a} = \frac{\sin B}{b}$.

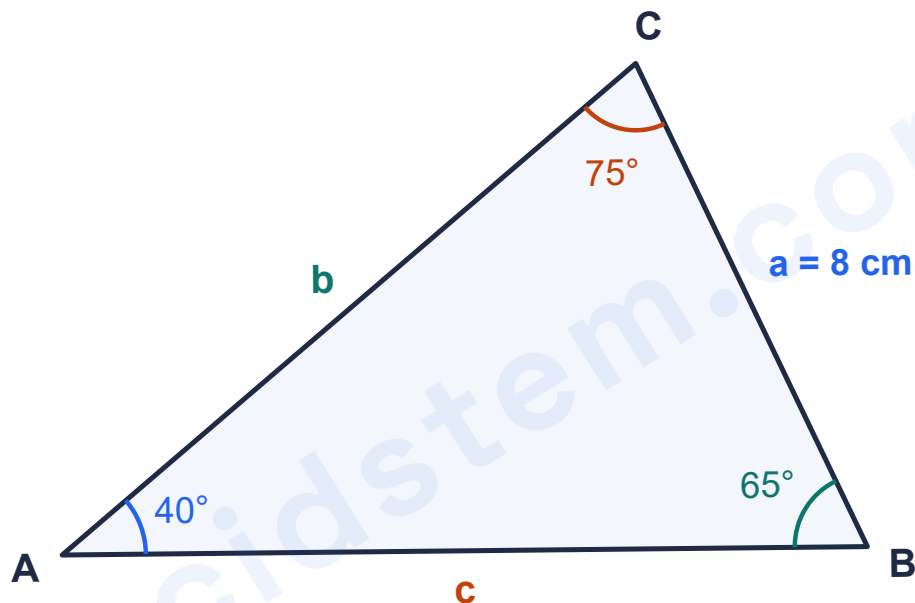


Fig. 1: triangle ABC drawn to scale with $A = 40^\circ$, $B = 65^\circ$, $C = 75^\circ$ and $a = 8$ cm. Each side is named after the angle opposite it: $b = 11.3$ cm and $c = 12.0$ cm

Worked example: Finding a side

In Fig. 1, $A = 40^\circ$, $B = 65^\circ$ and $a = 8$ cm. Find b .

1. The known pair is $a = 8$ with $A = 40^\circ$. We want b , opposite $B = 65^\circ$.

$$2. \frac{b}{\sin 65^\circ} = \frac{8}{\sin 40^\circ}.$$

$$3. \text{ Multiply by } \sin 65^\circ: b = \frac{8 \sin 65^\circ}{\sin 40^\circ} = 11.279 \dots$$

Answer: $b = 11.3$ cm (3 s.f.)

Find a complete opposite pair first.

- ✓ **Label:** side a opposite angle A , b opposite B , c opposite C .
- ✓ **Pair:** find a side and its opposite angle (use 180° for the third angle if needed).
- ✓ **Side:** $b = \frac{a \sin B}{\sin A}$.
- ✓ **Angle:** $\sin B = \frac{b \sin A}{a}$; then ask whether $180^\circ - B$ also works.
- ✓ **Accuracy:** sides 3 s.f., angles 1 d.p.; keep full values between steps.

Watch out!

Pairing a side with an angle that is NOT opposite it. Write the triangle's letters on your sketch and check: side a must face angle A . Using $\sin 73^\circ$ where the opposite angle is really 55° is the commonest lost mark.

Always giving the calculator's acute angle. If the question says the angle is obtuse, or your diagram shows it is, the answer is 180° minus the calculator value.

Putting the bearing itself (040° or 300°) into the sine rule.

Bearings are measured from north; the sine rule needs the angles inside the triangle.

8 The cosine rule

Syllabus 6.5

$a^2 = b^2 + c^2 - 2bc \cos A$, where A is the angle between sides b and c . To find an angle from three sides, rearrange: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$. A negative cosine means an obtuse angle.

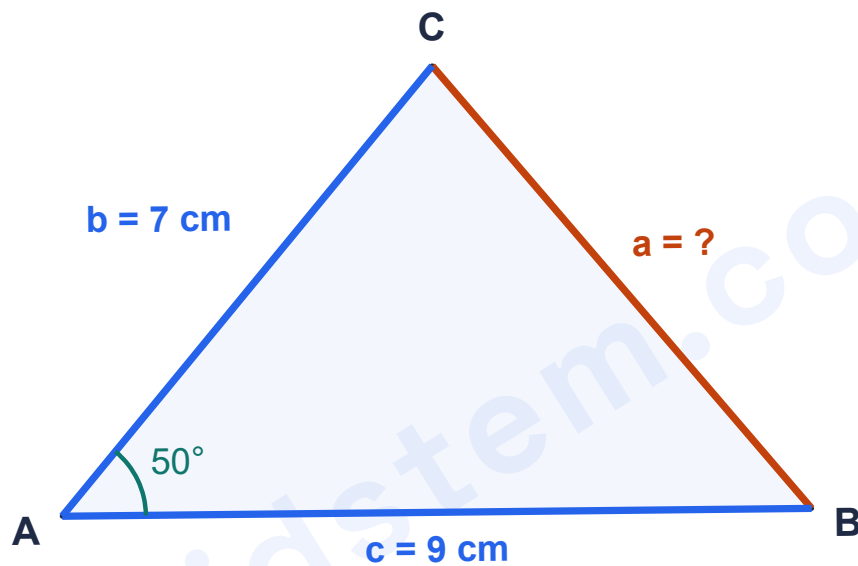


Fig. 1: two sides and the angle between them, drawn to scale ($b = 7 \text{ cm}$, $c = 9 \text{ cm}$, $A = 50^\circ$). The cosine rule gives the third side $a = 7.00 \text{ cm}$

Worked example: Finding the third side

In Fig. 1, $b = 7 \text{ cm}$, $c = 9 \text{ cm}$ and $A = 50^\circ$. Find a .

1. A is between sides b and c , so use the cosine rule: $a^2 = 7^2 + 9^2 - 2 \times 7 \times 9 \times \cos 50^\circ$.
2. $a^2 = 49 + 81 - 126 \cos 50^\circ = 130 - 80.99 \dots = 49.008 \dots$
3. $a = \sqrt{49.008 \dots} = 7.0006 \dots$

Answer: $a = 7.00 \text{ cm}$ (3 s.f.)

Side opposite the angle, then substitute carefully.

- ✓ **Side:** $a^2 = b^2 + c^2 - 2bc \cos A$, with A between b and c .
- ✓ **Angle:** $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, subtracting the side opposite A .
- ✓ **Obtuse:** a negative cosine means an obtuse angle; no ambiguity.
- ✓ **Choose:** complete opposite pair, sine rule; two sides and included angle or three sides, cosine rule.
- ✓ **Accuracy:** keep full values; sides 3 s.f., angles 1 d.p.

Watch out!

Typing $b^2 + c^2 - 2bc$ first and then multiplying by $\cos A$. For $p = 8$, $q = 11$, $R = 72^\circ$ that gives $\sqrt{9 \cos 72^\circ} = 1.67$ instead of 11.4. The $\cos A$ multiplies only the $2bc$ term.

Subtracting the wrong side. The side you subtract must be the one **OPPOSITE** the angle you want. Subtracting 5 instead of 9 above gives the smallest angle, not the largest.

Using the rounded $a = 10.4$ in the sine rule. Keep $10.436 \dots$ in the calculator, or the angle may come out wrong in the first decimal place.

9 Area of a triangle: half $ab \sin C$

Syllabus 6.5

Area = $\frac{1}{2}ab \sin C$, where C is the angle BETWEEN the sides a and b . It works because the perpendicular height onto side a is $b \sin C$.

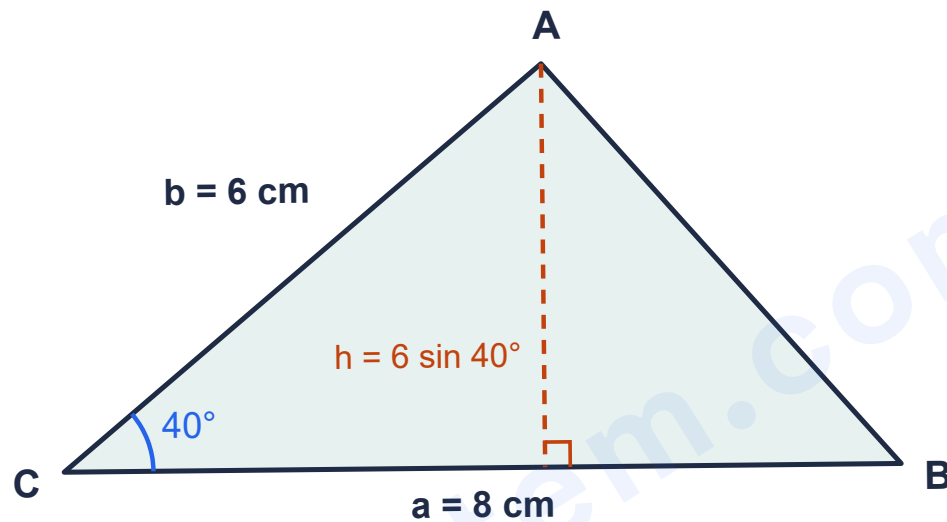


Fig. 1: drawn to scale with $a = 8$ cm, $b = 6$ cm and $C = 40^\circ$. The height is $h = 6 \sin 40^\circ = 3.86$ cm, so the area is $\frac{1}{2} \times 8 \times 6 \sin 40^\circ = 15.4$ cm²

Worked example: Finding the area

In Fig. 1, $a = 8$ cm, $b = 6$ cm and $C = 40^\circ$. Find the area of the triangle.

1. C is the angle between a and b , so the formula applies directly.
2. Area = $\frac{1}{2} \times 8 \times 6 \times \sin 40^\circ = 24 \sin 40^\circ = 15.426 \dots$

Answer: 15.4 cm² (3 s.f.)

Two sides, the angle between them, and a half.

- ✓ **Formula:** $\text{Area} = \frac{1}{2}ab \sin C$, with C between a and b .
- ✓ **Obtuse angles:** work exactly the same way; $\sin C$ is still positive.
- ✓ **Backwards:** $\sin C = \frac{2 \times \text{Area}}{ab}$; consider $180^\circ - C$ as well.
- ✓ **Three sides:** cosine rule for an angle first.
- ✓ **Segment:** sector area minus triangle area, both using the angle at the centre.

Watch out!

Using an angle that is not between the two sides. If you know sides 9 cm and 12 cm and an angle opposite one of them, first find the included angle (with the sine rule), then use the formula.

Forgetting the $\frac{1}{2}$ when working backwards. With area 30 cm^2 and sides 8 and 10, $\sin C = \frac{30}{80}$ gives 22.0° , which is wrong. The half becomes a 2 when you rearrange:
$$\sin C = \frac{2 \times \text{Area}}{ab}$$

In a segment problem, using the chord length as a side in $\frac{1}{2}ab \sin C$. The two sides of the triangle are the RADII, and C is the angle at the centre.

10 Pythagoras and trigonometry in 3D

Syllabus 6.6

Work in flat right-angled triangles. The space diagonal of a cuboid is $\sqrt{l^2 + w^2 + h^2}$. The angle between a line and a plane is the angle between the line and its projection (its 'shadow') on the plane, found in the right-angled triangle formed by dropping a perpendicular to the plane.

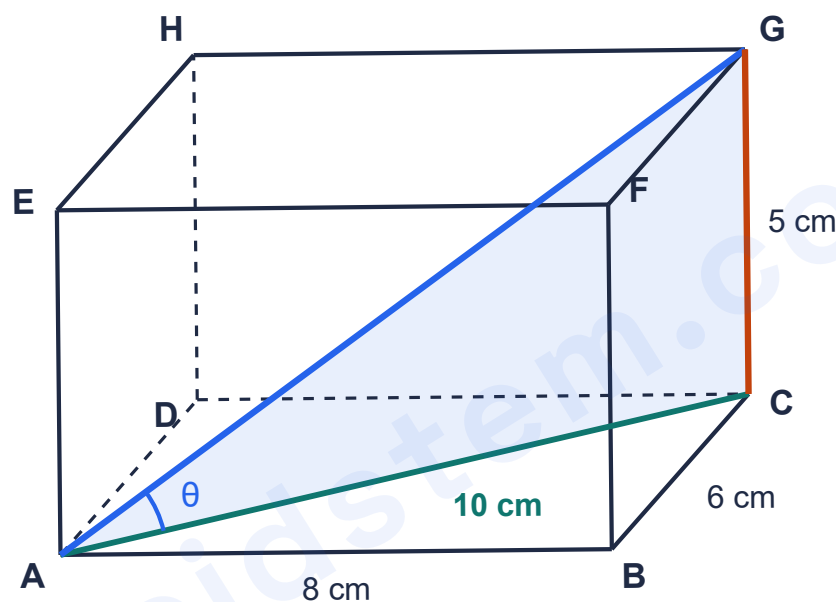


Fig. 1: a cuboid 8 cm by 6 cm by 5 cm. The shaded triangle ACG is right-angled at C : $AC = 10$ cm lies in the base, $CG = 5$ cm is vertical, and θ is the angle between the diagonal AG and the base $ABCD$

Worked example: The space diagonal

In Fig. 1, $AB = 8$ cm, $BC = 6$ cm and $CG = 5$ cm. Find the length of AG .

1. Triangle ABC (flat on the base, right angle at B): $AC^2 = 8^2 + 6^2 = 100$, so $AC = 10$ cm.
2. Triangle ACG (upright, right angle at C): $AG^2 = 10^2 + 5^2 = 125$.
3. $AG = \sqrt{125} = 11.180 \dots$

Answer: $AG = 11.2$ cm (3 s.f.). Check: $\sqrt{8^2 + 6^2 + 5^2} = \sqrt{125}$.

One flat right-angled triangle at a time.

- ✓ **Label** the solid and mark every right angle (vertical meets horizontal at 90°).
- ✓ **Base first:** find the diagonal or distance in the base with Pythagoras.
- ✓ **Then upright:** use the vertical triangle for the slant length or the angle.
- ✓ **Line and plane:** drop a perpendicular; the angle is between the line and its projection.
- ✓ **Accuracy:** keep surds or full calculator values until the end.

Watch out!

Using half the side instead of half the **DIAGONAL** for a pyramid's sloping edge. M is the centre of the square, so AM is half of AC , not half of AB . (Half the side is correct only for the slant height up the middle of a face.)

Using the wrong line in the base. The angle between AG and the base is **NOT** angle GAB : AB is not the shadow of AG . Using $\tan^{-1}(5/8) = 32.0^\circ$ is the classic wrong answer.

Measuring the angle of elevation from a ground line that does not lead to the foot of the pole. The angle of elevation of P from C lives in the vertical triangle PAC , so you need AC , the line from C to the foot A .